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A graded Gersten–Witt complex for schemes with a dualizing complex and the Chow group

Stefan Gille

Mathematisches Institut, Universität München, Theresienstrasse 39, 80333 München, Germany

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Abstract

We construct for any scheme X with a dualizing complex $I_\bullet$ a Gersten–Witt complex $\mathrm{GW}(X, I_\bullet)$ and show that the differential of this complex respects the filtration by the powers of the fundamental ideal. To prove this we introduce second residue maps for one-dimensional local domains which have a dualizing complex. This residue maps generalize the classical second residue morphisms for discrete valuation rings. For the cohomology of the quotient complexes $\mathrm{GrGW}_p(X, I_\bullet)$ of this filtration we prove $H^p(\mathrm{GrGW}_p(X, I_\bullet)) \simeq \mathrm{CH}^p(X, \mu_I)/2$, where μ_I is the codimension function of the dualizing complex $I_\bullet$ and $\mathrm{CH}^p(X, \mu_I)$ denotes the Chow group of μ_I-codimension p-cycles modulo rational equivalence.

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0. Introduction

Let X be a scheme with dualizing complex $I_\bullet$. Associated with $I_\bullet$ we have a codimension function $\mu_I : X \longrightarrow \mathbb{Z}$; see [17, Chap. V, Section 7]. The codimension function μ_I defines a filtration on the bounded derived category of quasi-coherent $\mathcal{O}_X$-modules. From this filtration we get a spectral sequence $E_1^{p,q}(X, I_\bullet)$, the Gersten–Witt spectral sequence, which converges to the coherent Witt theory of X:

$$E_1^{p,q}(X, I_\bullet) \Longrightarrow \tilde{W}^{p+q}(X, I_\bullet), \tag{1}$$

which generalizes the Gersten–Witt spectral sequence of a regular scheme introduced by Balmer and Walter [6].

The 0-th line of the spectral sequence (1) is isomorphic to a Gersten–Witt complex:

$$\bigoplus_{\mu_I(x)=m} W(k(x)) \xrightarrow{d^0} \bigoplus_{\mu_I(x)=m+1} W(k(x)) \xrightarrow{d^1} \bigoplus_{\mu_I(x)=m+2} W(k(x)) \xrightarrow{d^2} \cdots, \tag{2}$$

where $m = \min \mu_I$, and $W(k(x))$ denotes the Witt group of the residue field of x.

E-mail address: gille@mathematik.uni-muenchen.de.

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For $d \geq 0$ let $I^d(k) \subseteq W(k)$ be the d-th power of the fundamental ideal of even dimensional forms, and set $I^d(k) = W(k)$ for $d < 0$. We show that the differential of the Gersten–Witt complex (2) respects the filtration by the powers of the fundamental ideal as follows:

$$d^p \left(\bigoplus_{\mu_I(x)=p} I^d(k(x)) \right) \subseteq \bigoplus_{\mu_I(x)=p+1} I^{d-1}(k(x)). \tag{3}$$

We get a filtration, the filtration by the powers of the fundamental ideal, of the Gersten–Witt complex:

$$\mathrm{GW}(X, I_\bullet) = F^0\mathrm{GW}(X, I_\bullet) \supseteq F^1\mathrm{GW}(X, I_\bullet) \supseteq F^2\mathrm{GW}(X, I_\bullet) \supseteq \cdots,$$

where $F^d\mathrm{GW}(X, I_\bullet)$ is the following complex:

$$\bigoplus_{\mu_I(x)=m} I^d(k(x)) \xrightarrow{d^0} \bigoplus_{\mu_I(x)=m+1} I^{d-1}(k(x)) \xrightarrow{d^1} \bigoplus_{\mu_I(x)=m+2} I^{d-2}(k(x)) \xrightarrow{d^2} \cdots.$$

Associated with this filtration is a spectral sequence, the graded Gersten–Witt spectral sequence:

$$\mathrm{GrE}_1^{p,q}(X, I_\bullet) := H^{p+q}(\mathrm{GrGW}_p(X, I_\bullet)),$$

where the complex $\mathrm{GrGW}_p(X, I_\bullet)$ is the p-th quotient of this filtration:

$$\mathrm{GrGW}_p(X, I_\bullet) := F^p\mathrm{GW}(X, I_\bullet)/F^{p+1}\mathrm{GW}(X, I_\bullet). \tag{4}$$

We call this complex the graded Gersten–Witt complex in the following.

However, the filtration on the Gersten–Witt complex is not always finite (e.g. if X has a real point), and so in general we do not know much about convergence. But it is an easy consequence of the Milnor conjecture that it converges to the cohomology of the Gersten–Witt complex if X is a scheme of finite type over a field of finite cohomological 2-dimension (note that any scheme which is of finite type over a field has a dualizing complex).

Our main result about the graded Gersten–Witt complex $\mathrm{GrGW}_p(X, I_\bullet)$ is a Bloch–Ogus-type theorem [7]. We introduce for this the Chow group $\mathrm{CH}^p(X, \mu_I)$ of μ_I-codimension p-cycles modulo rational equivalence. This group is defined as the usual Chow group of codimension p-cycles modulo rational equivalence, except that we replace codimension by the codimension function μ_I of the dualizing complex in question. We prove then (see Theorem 9.2):

Theorem 0.1. *There is an isomorphism*

$$\mathfrak{e}^p_{X,I} : \mathrm{CH}^p(X, \mu_I) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\simeq} H^p(\mathrm{GrGW}_p(X, I_\bullet))$$

for all schemes X with dualizing complex $I_\bullet$.

If X is a Cohen–Macaulay scheme with dualizing sheaf Ω then a finite injective resolution $I_\bullet$ of the dualizing sheaf is a dualizing complex and the codimension function of this dualizing complex is the usual codimension. In particular we have then $\mathrm{CH}^p(X, \mu_I) = \mathrm{CH}^p(X)$, the Chow group of codimension p-cycles modulo rational equivalence. Hence by Theorem 0.1 above, the differential of the graded Gersten–Witt spectral sequence $\mathrm{GrE}_1^{p,0}(X, I_\bullet) \longrightarrow \mathrm{GrE}_1^{p+1,0}(X, I_\bullet)$ induces a homomorphism of usual Chow groups modulo 2:

$$S^p_{X,\Omega} : \mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{CH}^{p+1}(X) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

If $X = \operatorname{Spec} A$ is an affine Cohen–Macaulay scheme this homomorphism is given by (see Theorem 9.5):

Theorem 0.2. *Let A be a Cohen–Macaulay ring with canonical module Ω, and $P \in \operatorname{Spec} A$ be a prime ideal of height p. Then*

$$S^p_{A,\Omega}([\operatorname{Spec} A/P]) = \sum_{\mathrm{ht}\, Q=p+1,\ Q\supset P} \mathrm{length}\left(\mathrm{Coker}(A/P \xrightarrow{\iota} \mathrm{Ext}^p_A(A/P, \Omega))_Q\right) [A/Q],$$

(modulo $2\mathrm{CH}^{p+1}(A)$), where ι is an arbitrary embedding $A/P \hookrightarrow \mathrm{Ext}^p_A(A/P, \Omega)$. In particular if A/P is regular then $S^p_{A,\Omega}([\operatorname{Spec} A/P]) = 0$.

We give some comments on the proof of (3). For the verification of this inclusion we introduce generalized second residue maps for any local one-dimensional domain $(A, \mathfrak{m}, k)$ with dualizing complex:

$$d^0_{(\iota_{(0)},\iota_{\mathfrak{m}})} : W(K) \longrightarrow W(k)$$

(K is the quotient field of A). Since A has a dualizing complex it has a canonical module Ω. The residue map $d^0_{(\iota_{(0)},\iota_{\mathfrak{m}})}$ depends on an isomorphism $\iota_{(0)} : K \xrightarrow{\simeq} \Omega_{(0)}$ as well as on an embedding $\iota_{\mathfrak{m}}$ of the residue field k into its A-injective hull. If A is regular, i.e. a discrete valuation ring, we prove that $d^0_{(\iota_{(0)},\iota_{\mathfrak{m}})}$ is equal to a classical second residue map associated with some uniformizer. Our main result about this generalized second residue map is Theorem 6.6:

$$d^0_{(\iota_{(0)},\iota_{\mathfrak{m}})}(I^d(K)) \subseteq I^{d-1}(k). \tag{5}$$

Note that we cannot reduce the proof of (5) via normalization to the same result for the classical second residue map, in which case this is well known (cf. [1, Satz 3.1]), since in our more general situation the normalization morphism is not always finite. Instead we verify (5) by induction on d using that $d^0_{(\iota_{(0)},\iota_{\mathfrak{m}})}$ is $W(A)$-linear and the following fact: If $\alpha \in W(K)$ is a d-Pfister form with $d \geq 2$, then there is a finite extension $K \supset B \supseteq A$ such that $\alpha \in \operatorname{Im}(W(B) \times I^{d-1}(K) \longrightarrow W(K))$.

It turns out that as for the classical second residue morphism (cf. [1]), the induced morphism

$$I^d(K)/I^{d+1}(K) \longrightarrow I^{d-1}(k)/I^d(k)$$

does not depend on choices. As a consequence the same is true for the graded Gersten–Witt complex (4) as well, something which is pretty wrong for the "usual" Gersten–Witt complex (2).

There is another construction of a graded Gersten–Witt complex due to Rost [29]. By results of Arason [1] the functor $k \longmapsto \bigoplus_{d\geq 0} I^d(k)/I^{d+1}(k)$ from the category of fields to $\mathbb{Z}$-graded abelian groups is a cycle module in the sense of Rost [29]. Let X be a scheme which is essentially of finite type over a field k, i.e. X is of finite type over k or a localization of such a scheme. Giving such a scheme X, Rost [29, Sect. 5] constructs a so-called cycle cocomplex

$$\bigoplus_{x\in X^{(0)}} \bar{I}^d(k(x)) \longrightarrow \bigoplus_{x\in X^{(1)}} \bar{I}^{d-1}(k(x)) \longrightarrow \bigoplus_{x\in X^{(2)}} \bar{I}^{d-2}(k(x)) \longrightarrow \cdots, \tag{6}$$

where $X^{(p)} \subset X$ denotes the set of codimension p points and $\bar{I}^d(k(x))$ is the quotient $I^d(k(x))/I^{d+1}(k(x))$. If the scheme X has a dualizing complex $I_\bullet$ such that the codimension function of $I_\bullet$ is the usual codimension, i.e. $\mu_I(x) = \dim \mathcal{O}_{X,x}$ for all $x \in X$, then this cycle cocomplex (6) coincides with the graded Gersten–Witt complex $\mathrm{GrGW}_d(X, I_\bullet)$; see Theorem 7.7. Examples of such schemes are irreducible varieties or smooth (over a field) schemes. In the latter case any injective resolution of the structure sheaf is such a dualizing complex.

A graded Gersten–Witt spectral sequence appears also in the unpublished work of Pardon [27], but only for Cohen–Macaulay schemes which are of finite type over a field or which are a localization of such a finite type scheme. In this work Pardon constructs a Gersten–Witt complex, too, but not a Gersten–Witt spectral sequence, like the Balmer–Walter one [6]. Note that it is unknown (at least there does not exist a written proof) whether Pardon's Gersten–Witt complex and the Balmer–Walter one coincide or not.

Using the Milnor conjecture Pardon [27] has also proven the analog of our Theorem 0.1 for his graded Gersten–Witt spectral sequence for smooth varieties. Although the Milnor conjecture is not a conjecture any longer we would like to point out that our proof of Theorem 0.2 does not use the Milnor conjecture.

Notation and conventions. Let $\mathcal{E}$ be an exact category. Then the symbol $D^b(\mathcal{E})$ denotes the bounded derived category of $\mathcal{E}$. As usual in triangular Witt theory we use homological complexes. In particular the shifted complex $K[1]_\bullet$ of $K_\bullet \in D^b(\mathcal{E})$ is given by $K[1]_i = K_{i-1}$ and $d_i^{K[1]} = -d_{i-1}^K$.

Let X be a scheme with structure bundle $\mathcal{O}_X$. Then $\mathcal{M}(X)$ denotes the category of quasi-coherent $\mathcal{O}_X$-modules, $\mathcal{M}_{coh}(X)$ the category of coherent $\mathcal{O}_X$-modules, and $\mathcal{P}(X)$ the category of locally free $\mathcal{O}_X$-modules of finite rank. The symbol $D^b_{coh}(\mathcal{M}(X))$ denotes the full subcategory of $D^b(\mathcal{M}(X))$ consisting of complexes whose homology modules are in $\mathcal{M}_{coh}(X)$. If $X = \operatorname{Spec} A$ we use the symbols $\mathcal{M}(A), \mathcal{M}_{coh}(A), \ldots$.

The support of the $\mathcal{O}_X$-module M is the set

$$\operatorname{supp} M := \{x \in X | M_x \neq 0\},$$

and the (homological) support of the complex $M_\bullet \in D^b(\mathcal{M}(X))$ is the set

$$\operatorname{supp} M_\bullet := \{x \in X \mid (M_\bullet)_x \neq 0 \text{ in } D^b(\mathcal{M}(\mathcal{O}_{X,x}))\} = \bigcup_{i\in\mathbb{Z}} \operatorname{supp} H_i(M_\bullet).$$

We use the following sign convention for the total Hom-complex. Let $(M_\bullet, d_\bullet^M)$ and $(N_\bullet, d_\bullet^N)$ be two complexes of $\mathcal{O}_X$-modules. Then the differential

$$\mathcal{H}om_{\mathcal{O}_X}(M_\bullet, N_\bullet)_l \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(M_\bullet, N_\bullet)_{l-1}$$

is given by

$$f \longmapsto f \cdot d^M + (-1)^{l+1} d^N \cdot f. \tag{7}$$

This choice of sign is such that $\mathcal{H}om_{\mathcal{O}_X}(M_\bullet[1], N_\bullet)$ is equal to $\mathcal{H}om_{\mathcal{O}_X}(M_\bullet, N_\bullet)[-1]$, and not only naturally isomorphic.

We assume throughout that all schemes are noetherian and have 1/2 *in their global section. All rings are assumed to be commutative with* 1.

1. Injective hulls and dualizing complexes

1.1. For the convenience of our reader we collect here some results about injective hulls and dualizing complexes which we will use throughout.

Let X be a scheme with structure sheaf $\mathcal{O}_X$, and M a coherent $\mathcal{O}_X$-module. We say that an injective $\mathcal{O}_X$-module I is an injective hull of M if there is a monomorphism $M \hookrightarrow I$ and for any $\mathcal{O}_X$-submodule $0 \neq N \subseteq I$ the natural homomorphism of $\mathcal{O}_X$-modules $I \longrightarrow I/M \oplus I/N$ has non-zero kernel, which is the intersection $M \cap N$. The latter means that I is an essential extension of M. Note that two injective hulls of M are isomorphic, although not canonically. Nevertheless we will speak (as usual) in the following of *the injective hull* of the $\mathcal{O}_X$-module M and denote it $\mathrm{E}_{\mathcal{O}_X}(M)$. Since X is noetherian the injective hull of a quasi-coherent $\mathcal{O}_X$-module exists and is a quasi-coherent $\mathcal{O}_X$-module; see [17, Chap. II, Section 7]. The following result shows that I is an injective hull of M if and only if I_x is an injective hull of M_x for all $x \in X$.

Lemma 1.2. *Let X be a scheme and κ an open immersion $U \subseteq X$, or an inclusion* $\operatorname{Spec} \mathcal{O}_{X,x} \hookrightarrow X$. *Let $M \subseteq V$ be an essential extension of quasi-coherent $\mathcal{O}_X$-modules. Then $\kappa^*(M) \longrightarrow \kappa^*(V)$ is an essential extension of $\mathcal{O}_U$-modules. In particular $\kappa^*(\mathrm{E}_{\mathcal{O}_X}(M))$ is an injective hull of $\kappa^*(M)$.*

Proof. A proof in the affine case can be found in [8, Lem. 3.2.5], or in [32, Prop. 3.3.2]. The latter can be adapted to the case of an open immersion $\kappa : U \subseteq X$ as follows.

It is enough to check that any coherent submodule of $V|_U$, which is non-zero, has non-trivial intersection with $M|_U$. By [16, Cor. 6.9.3] any such submodule is the restriction of a coherent $\mathcal{O}_X$-module $W \subseteq V$. Let $W \subseteq V$ be a coherent $\mathcal{O}_X$-module with $W|_U \neq 0$. Then we have $\mathcal{J}^s W \neq 0$ and hence $\mathcal{J}^s W \cap M \neq 0$ for all $s \geq 0$, where $\mathcal{J}$ is the ideal sheaf which defines the closed complement of U. If now $0 = W|_U \cap M|_U = (W \cap M)|_U$ then there exists $s \geq 0$ such that $\mathcal{J}^s(W \cap M) = 0$. But by the Artin–Rees Lemma (e.g. [22, Thm. 8.5]) applied to the pair of coherent $\mathcal{O}_X$-modules $W \cap M \subseteq W$ there is a natural number r_0 such that

$$0 \neq (\mathcal{J}^r W) \cap M = (\mathcal{J}^r W) \cap (W \cap M) = \mathcal{J}^{r-r_0}((\mathcal{J}^{r_0} W) \cap M)$$

for all $r \geq r_0$, a contradiction. □

Corollary 1.3. *Let X be a scheme and $M \subseteq V$ quasi-coherent $\mathcal{O}_X$-modules. Then V is an essential extension of M if and only if $M_x \subseteq V_x$ is an essential extension of $\mathcal{O}_{X,x}$-modules for all $x \in X$. In particular V is an injective hull of M if and only if V_x is an injective hull of M_x for all $x \in X$ (considering the zero module as the injective hull of the zero module).*

Proof. The first part is an obvious consequence of the lemma above and the second part follows from this and the fact that since X is noetherian an $\mathcal{O}_X$-module is injective if and only if it is locally injective; see [17, Chap. II, Prop. 7.18]. □

In the affine case we have the following structure theorem for injective modules due to Matlis [21]; see also [8, Thms. 3.2.6 and 3.2.8].

Theorem 1.4. *Let A be a ring and I an injective A-module. Then:*

(i) *The module I is indecomposable if and only if $I = \mathrm{E}_A(A/P)$ for some $P \in \operatorname{Spec} A$.*
(ii) *For all $P, Q \in \operatorname{Spec} A$ we have $\mathrm{E}_A(A/P) \simeq \mathrm{E}_A(A/Q)$ if and only if $P = Q$.*
(iii) *There is an unique decomposition of I into indecomposable injective modules:*

$$I \simeq \bigoplus_{P \in \operatorname{Spec} A} \mathrm{E}_A(A/P)^{m(P,I)}.$$

In particular, the maybe infinite numbers $m(P, I)$ depend only on I and the prime ideal P.

We will have need for the following easy but useful lemma whose proof is left to the reader.

Lemma 1.5. *Let $(A, \mathfrak{m}, k)$ be a local ring and $\iota_i : k \hookrightarrow \mathrm{E}_A(k)$, $i = 1, 2$, be two injective homomorphisms of the residue field k into its injective hull. Then ι_1 and ι_2 are essential extensions, and there exists an $a \in A^\times = A \setminus \mathfrak{m}$ such that $\iota_2 = a \cdot \iota_1$.*

1.6. *Dualizing complexes.* Let X be a scheme and $I_\bullet \in D^b_{coh}(\mathcal{M}(X))$ a complex of injective $\mathcal{O}_X$-modules:

$$\cdots \longrightarrow 0 \longrightarrow I_{-m} \xrightarrow{d^I_{-m}} I_{-m-1} \xrightarrow{d^I_{-m-1}} \cdots \xrightarrow{d^I_{-n+1}} I_{-n} \longrightarrow 0 \longrightarrow \cdots \tag{8}$$

$(n > m)$. We have a natural morphism of complexes:

$$\varpi^I_M : M_\bullet \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(M_\bullet, I_\bullet), I_\bullet)$$

which is essentially the evaluation map. However, there are some signs involved which are important for us, so let us fix them. The (r, s)-component of

$$(\varpi^I_M)_l : M_l \longrightarrow \bigoplus_{r\in\mathbb{Z}} \bigoplus_{s\in\mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(M_{l+r-s}, I_{-s}), I_{-r})$$

is $(-1)^{\frac{s(s+1)}{2}}$ times the evaluation morphism if $r = s$, and 0 otherwise.

Definition 1.7. A complex $I_\bullet \in D^b_{coh}(\mathcal{M}(X))$ of injective modules is called a *dualizing complex* of X if ϖ^I_M is an isomorphism for all $M_\bullet \in D^b_{coh}(\mathcal{M}(X))$. If moreover I_{-r} is an essential extension of $\operatorname{Ker} d^I_{-r}$ for all $r \in \mathbb{Z}$ we say that $I_\bullet$ is a *minimal dualizing complex*.

Example 1.8. (i) Any scheme which is essentially of finite type (i.e. of finite type or a localization of a finite type scheme) over a regular scheme of finite Krull dimension has a dualizing complex; see [17, Chap. V, Section 10].
(ii) Assume X is a Cohen–Macaulay scheme with a dualizing module (also called a canonical module or sheaf) Ω. Then any finite injective resolution

$$I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow I_{-n} \longrightarrow 0$$

of Ω considered as an element of $D^b_{coh}(\mathcal{M}(X))$ with I_r in degree r is a dualizing complex of X. If $I_\bullet$ is a minimal injective resolution then it is a minimal dualizing complex.

Proposition 1.9. *Any dualizing complex is quasi-isomorphic to a minimal dualizing complex.*

Proof. Let $I_\bullet : \cdots \longrightarrow I_s \xrightarrow{d^I_s} I_{s-1} \longrightarrow \cdots$ be a dualizing complex of X. We construct inductively a minimal dualizing complex $I'_\bullet$ and a quasi-isomorphism $f = f_\bullet : I_\bullet \longrightarrow I'_\bullet$, i.e. f_r induces an isomorphism $H_r(I_\bullet) \xrightarrow{\simeq} H_r(I'_\bullet)$.

We can assume $I_0 \neq 0$ and $I_r = 0$ for $r \geq 1$. Let $I'_r = 0$ for $r \geq 1$ and I'_0 be an injective hull of $\operatorname{Ker} d^I_0$. Since I'_0 is injective, the monomorphism $\operatorname{Ker} d^I_0 \hookrightarrow I'_0$ extends to a homomorphism $f_0 : I_0 \longrightarrow I'_0$.

Assume we have constructed everything in degrees $\geq -(s-1)$ with $s \geq 1$. Let then

$$M_{-s} := \operatorname{Ker} d^I_{-s} \bigoplus_{I_{-(s-1)}} (I'_{-(s-1)}/\operatorname{Im} d^{I'}_{-(s-2)})$$

(coproduct), and let I'_{-s} be the injective hull of M_{-s}. In particular, I'_{-s} is an essential extension of M_{-s}. As above, the composition $\operatorname{Ker} d^I_{-s} \longrightarrow M_{-s} \longrightarrow I'_{-s}$ extends to a homomorphism $f_{-s} : I_{-s} \longrightarrow I'_{-s}$.

It is straightforward to check that $f : I_\bullet \longrightarrow I'_\bullet$ is a quasi-isomorphism. To see that I'_{-s} is an essential extension of $\operatorname{Ker} d^{I'}_{-s}$ note that $M_{-s} \longrightarrow I'_{-s} \xrightarrow{d^{I'}_{-s}} I'_{-s-1}$ is the zero morphism, since M_{-s} is generated by $\operatorname{Im} d^{I'}_{-(s-1)}$ and $f_{-s}(\operatorname{Ker} d^I_{-s})$. $\square$

The first part of the following proposition follows easily from the definition and Lemma 1.2.

Proposition 1.10. *Let X be a scheme with dualizing complex $I_\bullet$.*

(i) *If κ is an open embedding $U \subseteq X$ or an inclusion $\operatorname{Spec} \mathcal{O}_{X,x} \hookrightarrow X$, then $\kappa^*(I_\bullet)$ is a dualizing complex of U, which is minimal if $I_\bullet$ is minimal.*
(ii) *If $X = \operatorname{Spec} A$ is affine, $J \subset A$ an ideal, and $\pi : A \longrightarrow A/J$ the quotient map, then $\pi^\natural(I_\bullet) := \operatorname{Hom}_A(A/J, I_\bullet)$ is a dualizing complex of A/J. It is minimal if $I_\bullet$ is a minimal dualizing complex.*

Proof. Only (ii) is left to prove. The assertion that $\pi^\natural(I_\bullet)$ is a dualizing complex is a special case of [17, Prop. 2.4]; see also [13, Sect. 4]. To see that it is minimal, just note that $\operatorname{Hom}_A(A/J, I_{-r}) \simeq \{x \in I_{-r} \mid J \cdot x = 0\}$. $\square$

Dualizing complexes are unique on connected components:

Theorem 1.11. *Let X be a connected scheme, and $I_\bullet$ and $I'_\bullet$ two dualizing complexes of X. Then there exists an integer $m \in \mathbb{Z}$ and a line bundle $\mathcal{L}$ over X such that $I'_\bullet \simeq I_\bullet[m] \otimes_{\mathcal{O}_X} \mathcal{L}$ in $D^b_{coh}(\mathcal{M}(X))$.*

Proof. See [17, Chap. V, Thm. 3.1]. $\square$

The following lemma is a consequence of this fact; see [17, Chap. V, Prop. 3.4].

Lemma 1.12. *Let $(A, \mathfrak{m}, k)$ be a local ring with dualizing complex $I_\bullet$. Then there exists an integer d such that*

$$H_i(\operatorname{Hom}_A(k, I_\bullet)) \simeq \begin{cases} k & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Therefore we have the following well defined function $\mu_I : X \longrightarrow \mathbb{Z}$ for any scheme X with dualizing complex $I_\bullet$:

$$\mu_I(x) := -\min\{i \in \mathbb{Z} | H_i(\operatorname{Hom}_{\mathcal{O}_{X,x}}(k(x), (I_\bullet)_x) \neq 0\}$$

(for the "−"-sign: recall that we use homological complexes). We call this function the *codimension function* of the dualizing complex $I_\bullet$.

Example 1.13. If X is a Cohen–Macaulay scheme with dualizing module Ω, and $I_\bullet$ an injective resolution of Ω as in Example 1.8, then $\mu_I(x) = \dim \mathcal{O}_{X,x}$, i.e. μ_I is the usual codimension; cf. [8, Thm. 3.3.10].

The following lemma whose proof can be found in [17, Chap. V, Prop. 7.1] says that μ_I is a codimension function.

Lemma 1.14. *Let x, y be points of the scheme X. If y is an immediate specialization of x, i.e.*

(i) *x is in the closure of the point y, and*
(ii) *$\dim(\mathcal{O}_{X,x}/\kappa^{-1}(y)) = 1$, where $\kappa : \operatorname{Spec} \mathcal{O}_{X,x} \hookrightarrow X$ is the inclusion,*

then $\mu_I(x) = \mu_I(y) + 1$.

In particular, if X is irreducible then there exists an integer $m \in \mathbb{Z}$ such that $\mu_I(x) = \dim \mathcal{O}_{X,x} + m$ for all $x \in X$.

The value of the codimension function of a minimal dualizing complex at a point is the "position" of this point in this dualizing complex.

Theorem 1.15. *Let $I_\bullet$ be a minimal dualizing complex of the scheme X. Then for any open affine subset* $\operatorname{Spec} A \subseteq X$ *and all $r \in \mathbb{Z}$ we have*

$$I_{-r}|_{\operatorname{Spec} A} \simeq \bigoplus_{\mu_I(P)=r} \mathrm{E}_A(A/P).$$

Proof. By Proposition 1.10 the complex of A-modules $I_\bullet|_{\operatorname{Spec} A}$ is also a minimal dualizing complex, and so we can assume that $X = \operatorname{Spec} A$ is affine. We have

$$I_{-r} \simeq \bigoplus_{P \in \operatorname{Spec} A} \mathrm{E}_A(A/P)^{m(I_{-r},P)},$$

by Theorem 1.4, and so $\operatorname{Hom}_{A_Q}(k(Q), (I_{-r})_Q) \simeq k(Q)^{m(I_{-r},Q)}$. The complex $\operatorname{Hom}_{A_Q}(k(Q), (I_\bullet)_Q)$ is a minimal dualizing complex of $k(Q)$ by Proposition 1.10, and we have $H_i(\operatorname{Hom}_{A_Q}(k(Q), (I_\bullet)_Q) \neq 0$ if and only if $i = -\mu_I(Q)$ by Lemma 1.12. It follows that $m(I_{-r}, Q) = 0$ if $r \neq -\mu_I(Q)$ and $= 1$ otherwise, since a vector space is only an essential extension of itself. □

Remark 1.16. This means that a minimal dualizing complex is a residual complex; see [17, Chap. VI, Section 1].

2. Coherent Witt groups

2.1. Let X be a scheme with dualizing complex $I_\bullet$. Then $\mathfrak{D}_I := \mathcal{H}om_{\mathcal{O}_X}(-, I_\bullet)$ is a duality on the category $D^b_{coh}(\mathcal{M}(X))$ making it a triangulated category with 1-exact duality in the sense of Balmer [3]. More precisely the quadruple

$$(D^b_{coh}(\mathcal{M}(X)), \mathfrak{D}_I, 1, \varpi^I)$$

is a triangulated category with duality. We denote the i-th triangular Witt group of this category by $\tilde{W}^i(X, I_\bullet)$ and call it the i-th *coherent Witt group* of X with respect to $I_\bullet$. Recall the 4-periodicity: $\tilde{W}^i(X, I_\bullet) \simeq \tilde{W}^{i+4}(X, I_\bullet)$. The group $\tilde{W}^i(X, I_\bullet)$ classifies i-symmetric spaces up to neutral spaces. An i-symmetric space in $D^b_{coh}(\mathcal{M}(X))$ with respect to the duality $\mathfrak{D}_I$ is a pair $(M_\bullet, \phi)$, where $M_\bullet \in D^b_{coh}(\mathcal{M}(X))$ and $\phi : M_\bullet \longrightarrow \mathfrak{D}_I(M_\bullet)[i]$ is an isomorphism such that

$$\mathfrak{D}_I(\phi)[i] \cdot \varpi^I_M = (-1)^{\frac{i(i+1)}{2}} \phi.$$

The functor $\mathfrak{D}_I$ is also a duality on $D^b_{coh,Z}(\mathcal{M}(X)) \subseteq D^b_{coh}(\mathcal{M}(X))$, the subcategory of complexes with support in the closed subset $Z \subseteq X$. We denote its i-th triangular Witt group by $\tilde{W}^i_Z(X, I_\bullet)$, and call this group the i-th coherent Witt group of X *with support* in Z. If $X = \operatorname{Spec} A$ and Z is defined by the ideal $J \subseteq A$ we use the notation $\tilde{W}^i(A, I_\bullet)$ and $\tilde{W}^i_J(A, I_\bullet)$ instead of $\tilde{W}^i(X, I_\bullet)$ and $\tilde{W}^i_Z(X, I_\bullet)$, respectively.

2.2. There is a natural isomorphism

$$\mathfrak{sh}^s : \mathfrak{D}_{I[s]} \xrightarrow{\simeq} \mathfrak{D}_I[s],$$

which is given in degree l by $(-1)^{ls}\mathrm{id}$; cf. (7) for our sign conventions for the total $\mathcal{H}om_{\mathcal{O}_X}$-sheaf. This is a duality transformation for the identity functor, i.e.

(i) $\mathfrak{sh}^s_{M[1]} = (-1)^s \mathfrak{sh}^s_M[-1]$, and
(ii) the following diagram commutes:

$$\begin{tikzcd}
M_\bullet \arrow[r, "\varpi^{I[s]}_M"] \arrow[d, "(-1)^{\frac{s(s+1)}{2}} \varpi^I_M"'] & \mathfrak{D}_{I[s]}\mathfrak{D}_{I[s]}(M_\bullet) \arrow[d, "\mathfrak{sh}^s_{\mathfrak{D}_{I[s]}(M)}"] \\
\mathfrak{D}_I(\mathfrak{D}_I(M_\bullet)[s])[s] \arrow[r, "\mathfrak{D}_I(\mathfrak{sh}^s_M)[s]"] & \mathfrak{D}_I(\mathfrak{D}_{I[s]}(M_\bullet)[s]
\end{tikzcd}$$

for all $M_\bullet \in D^b_{coh}(\mathcal{M}(X))$; cf. [17, Def. 2.6]. In other words $(\mathrm{id}_{D^b_{coh}(\mathcal{M}(X))}, \mathfrak{sh}^s)$ is a duality preserving functor:

$$(D^b_{coh}(\mathcal{M}(X)), \mathfrak{D}_{I[s]}, 1, \varpi^{I[s]}) \longrightarrow (D^b_{coh}(\mathcal{M}(X)), \mathfrak{D}_I[s], (-1)^s, (-1)^{\frac{s(s+1)}{2}}\varpi^I). \tag{9}$$

Since $\mathrm{id}_{D^b_{coh}(\mathcal{M}(X))}$ is obviously an equivalence of categories we get from these remarks by [12, Thm. 2.7]:

Lemma 2.3. *The duality preserving functor $(\mathrm{id}_{D^b_{coh}(\mathcal{M}(X))}, \mathfrak{sh}^s)$ induces an isomorphism $\tilde{W}^i(X, I_\bullet[s]) \xrightarrow{\simeq} \tilde{W}^{i+s}(X, I_\bullet)$ for all $i \in \mathbb{Z}$.*

Example 2.4. If the scheme X is Gorenstein of finite Krull dimension then an injective resolution $I_\bullet$ (which is finite since we assume $\dim X < \infty$) of the structure bundle $\mathcal{O}_X$ is a dualizing complex of X. Any isomorphism $\mathcal{O}_X \xrightarrow{\simeq} I_\bullet$ in $D^b_{coh}(\mathcal{M}(X))$ makes then the canonical functor $D^b(\mathcal{P}(X)) \longrightarrow D^b_{coh}(\mathcal{M}(X))$ duality preserving and induces a homomorphism $W^i(X) \longrightarrow \tilde{W}^i(X)$ for all $i \in \mathbb{Z}$, where W^i denotes the derived Witt functor; see [4] for its definition. If X is moreover regular then it is an isomorphism. Note that this homomorphism is not natural.

Since by the main theorem of Balmer [4] the natural functor $\mathcal{P}(X) \longrightarrow D^b(\mathcal{P}(X))$ induces an isomorphism $W(X) \simeq W^0(X)$, we have for regular schemes X of finite Krull dimension a (not natural) isomorphism $W(X) \xrightarrow{\simeq} \tilde{W}^0(X)$ (here $W(X)$ denotes Knebusch's [19] Witt group of X).

3. Witt groups of finite length modules

3.1. Let $(A, \mathfrak{m}, k)$ be a local ring with minimal dualizing complex $I_\bullet$:

$$0 \longrightarrow I_{-m} \xrightarrow{d^I_{-m}} I_{-m-1} \xrightarrow{d^I_{-m-1}} \cdots \xrightarrow{d^I_{-n+2}} I_{-n+1} \xrightarrow{d^I_{-n+1}} I_{-n} \longrightarrow 0,$$

where $m = \min \mu_I$ and $n = \max \mu_I$. Note that since $I_\bullet$ is minimal we have $n = \mu_I(\mathfrak{m})$ and so $I_{-n} = \mathrm{E}_A(k) =: E$.

Let $\mathcal{M}_{fl}(A)$ be the category of finite length A-modules. The functor $M \longmapsto \mathrm{Hom}_A(M, E)$ is a duality on this category whose derived functor makes $D^b(\mathcal{M}_{fl}(A))$ a triangulated category with duality. We denote by $W^i(\mathcal{M}_{fl}(A), E)$ the i-th triangular Witt group of it. Since any k-vector space is naturally an A-module of finite length we have a natural functor $F : D^b(\mathcal{P}(k)) \longrightarrow D^b(\mathcal{M}_{fl}(A))$. To make this functor duality preserving we choose an embedding $\iota : k \hookrightarrow E$. This is an injective hull by Lemma 1.5. We have an isomorphism

$$\mathrm{Hom}_k(V, k) \xrightarrow{\simeq} \mathrm{Hom}_A(V, E) \qquad \alpha \longmapsto \iota \cdot \alpha$$

for all $V \in \mathcal{P}(k)$ by [12, Lem. 3.3], which extends to a natural isomorphism $\eta_\iota : \mathrm{Hom}_k(V_\bullet, k) \xrightarrow{\simeq} \mathrm{Hom}_A(V_\bullet, E)$, the duality transformation for F. Note that η_ι depends on ι. We get a homomorphism of Witt groups $W^i(k) \longrightarrow W^i(\mathcal{M}_{fl}(A), E)$ which is an isomorphism by [12, Thm. 3.7]. By [4] we know that $W^i(k) = 0$ if $4 \nmid i$ and the natural functor $\mathcal{P}(k) \longrightarrow D^b(\mathcal{P}(k))$ induces an isomorphism $W(k) \xrightarrow{\simeq} W^0(k)$, where $W^i(k)$ denotes the derived Witt group of k.

We have $\mathrm{Hom}_A(M, I_{-r}) = 0$ for all $m \le r \le n-1$ and all $M \in \mathcal{M}_{fl}(A)$ by [12, Lem. 3.3] and so we have a natural isomorphism

$$\mathrm{Hom}_A(M_\bullet, E) \xrightarrow{\simeq} \mathrm{Hom}_A(M_\bullet, I_\bullet)[n]$$

which is given in degree i by $(-1)^{in}\mathrm{id}_{\mathrm{Hom}_A(M_{-i}, E)}$. This is a duality transformation for the equivalence $D^b(\mathcal{M}_{fl}(A)) \longrightarrow D^b_{coh,\mathfrak{m}}(\mathcal{M}(A))$; see [12, Thm. 3.10]. Hence:

Theorem 3.2. *Let $(A, \mathfrak{m}, k)$ be a local ring with minimal dualizing complex $I_\bullet$. Set $n = \max \mu_I$. Then $\tilde{W}^{i+n}_{\mathfrak{m}}(A, I_\bullet) = 0$ for $i \not\equiv 0 \bmod 4$ and otherwise there is an isomorphism*

$$W(k) \xrightarrow{F^I_\iota} \tilde{W}^n_{\mathfrak{m}}(A, I_\bullet) \simeq \tilde{W}^{n+4i}_{\mathfrak{m}}(A, I_\bullet)$$

which depends on the choice of an injection $\iota : k \hookrightarrow \mathrm{E}_A(k) = I_{-n}$. The isomorphism F_ι is $W(A)$-linear.

Proof. Everything is proved except for the last assertion, which follows from the projection formula [13, Thm. 5.2], or more easily by direct verification. □

Remark 3.3. This isomorphism is compatible with the shift functor; see (9). More precisely, the following diagram commutes:

$$\begin{tikzcd} W(k) \arrow[r, "F_\iota^{I[s]}"] \arrow[d, "="'] & \tilde{W}^{n-s}_{\mathfrak{m}}(A, I_\bullet[s]) \arrow[d, "\simeq"', "(\mathrm{id},\mathfrak{sh}^s)_*"] \\ W(k) \arrow[r, "F_\iota^{I}"] & \tilde{W}^{n}_{\mathfrak{m}}(A, I_\bullet)\,. \end{tikzcd}$$

Notation 3.4. If the dualizing complex is clear from the context we will write F_ι instead of F_ι^I in the sequel. The same applies to other notation in the following.

3.5. The isomorphism F_ι is compatible with the transfer, too. Let $J \subset A$ be an ideal, and $\pi : A \longrightarrow A/J$ the quotient morphism. By Proposition 1.10 we know that $\pi^\natural(I_\bullet) = \mathrm{Hom}_A(A/J, I_\bullet)$ is a minimal dualizing complex of the ring A/J, and we have a morphism of complexes of A-modules $\zeta : \pi^\natural(I_\bullet) \longrightarrow I_\bullet$ which is given in degree $(-r)$ by

$$\zeta_{-r} \;:\; \mathrm{Hom}_A(A/J, I_{-r}) \longrightarrow I_{-r} \qquad \alpha \longmapsto (-1)^r \alpha(1 + J).$$

The morphism of complexes ζ induces an isomorphism of complexes $\mathfrak{D}_{\pi^\natural(I_\bullet)}(M_\bullet) \xrightarrow{\simeq} \mathfrak{D}_I(M_\bullet)$ for all $M_\bullet \in D^b_{coh}(\mathcal{M}(A/J))$ which is a duality transformation for the restriction of scalars functor $\pi_* : D^b_{coh}(\mathcal{M}(A/J)) \longrightarrow D^b_{coh}(\mathcal{M}(A))$. The induced morphism of Witt groups is the transfer morphism:

$$\mathrm{Tr}_{(A/J)/A} \;:\; \tilde{W}^i(A/J, \pi^\natural(I_\bullet)) \longrightarrow \tilde{W}^i(A, I_\bullet);$$

see [13, Sect. 4]. Note that ζ_{-n} is injective, and therefore gives an isomorphism $\pi^\natural(I_\bullet)_{-n} \xrightarrow{\simeq} \{x \in I_{-n} \mid J \cdot x = 0\} \simeq \mathrm{E}_{A/J}(k)$, i.e. we can identify $\pi^\natural(I_\bullet)_{-n}$ with an A/J-injective hull of k. It follows that there exists a monomorphism $\iota_J : k \longrightarrow \pi^\natural(I_\bullet)_{-n} = \mathrm{E}_{A/J}(k)$ such that $\iota = \zeta_{-n} \cdot \iota_J$. Since $\mu_{\pi^\natural(I)}(\mathfrak{m}/J) = \mu_I(\mathfrak{m}) = n$ we get

Proposition 3.6. *The following diagram commutes:*

$$\begin{tikzcd} W(k) \arrow[r, "F_{\iota_J}"] \arrow[d, "="'] & \tilde{W}^{n}_{\mathfrak{m}/J}(A/J, \pi^\natural(I_\bullet)) \arrow[d, "\mathrm{Tr}_{(A/J)/A}"] \\ W(k) \arrow[r, "F_\iota"] & \tilde{W}^{n}_{\mathfrak{m}}(A, I_\bullet)\,. \end{tikzcd}$$

3.7. In the proof of Proposition 6.10 we will need the following generalization of Theorem 3.2. Let A be a semilocal ring with minimal dualizing complex $I_\bullet$, $\mathfrak{m}$ a maximal ideal of A with $\mu_I(\mathfrak{m}) = n$, and $k = A/\mathfrak{m}$ the residue field of $\mathfrak{m}$. Let further $\iota : k \hookrightarrow E := \mathrm{E}_A(k)$ be an A-injective hull of k. Since we assume that $I_\bullet$ is minimal the module E appears only in $I_{-n} \simeq \bigoplus_{\mu_I(P)=n} \mathrm{E}_A(A/P)$ by Theorem 1.15. Hence we have

$$\mathrm{Hom}_A(k, I_{-r}) \;=\; \begin{cases} 0 & \text{if } r \neq n \\ \mathrm{Hom}_A(k, E) & \text{if } r = n \end{cases}$$

by [12, Lem. 3.3]. Let $\tau : k \longrightarrow I_{-n}$ be the composition of ι with the natural injection $E \subseteq I_{-n}$, and (V, ϕ) a symmetric k-space. Then

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "\tau\cdot\phi"] & 0 \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & \mathrm{Hom}_A(V, I_{-(n-1)}) \arrow[r] & \mathrm{Hom}_A(V, I_{-n}) \arrow[r] & \mathrm{Hom}_A(V, I_{-n-1}) \arrow[r] & \cdots \\
 & \deg 1 & \deg 0 & &
\end{tikzcd}$$

is an n-symmetric space in $D^b_{coh,\,\mathfrak{m}}(\mathcal{M}(A)) \subseteq$

$$D^{(n)}_{A,\mu_I} := \{M_\bullet \in \mathcal{M}_{coh}(A) \mid \mu_I(P) \geq n \text{ for all } P \in \operatorname{supp} M_\bullet\}.$$

This association gives us a homomorphism $F^{semi}_\iota : W(k) \longrightarrow W^n(D^{(n)}_{A,\mu_I}, I_\bullet)$, where $W^n(D^{(n)}_{A,\mu_I}, I_\bullet)$ denotes the n-th triangular Witt group of the triangulated category $D^{(n)}_{A,\mu_I}$ with respect to the duality $\mathfrak{D}_I = \mathcal{H}om_A(-, I_\bullet)$.

Proposition 3.8. *Let A be a semilocal ring with minimal dualizing complex $I_\bullet$. Denote by $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ the finitely many maximal ideals of A and by $k_i := A/\mathfrak{m}_i$ their respective residue fields. If $\mu_I(\mathfrak{m}_i) = n$ for all $1 \leq i \leq s$, then the homomorphism*

$$\bigoplus_{i=1}^{s} W(k_i) \xrightarrow{\sum F^{semi}_{\iota_i}} W^n(D^{(n)}_{A,\mu_I}, I_\bullet)$$

is an isomorphism, where $\iota_i : k_i \hookrightarrow \mathrm{E}_A(k_i) = \mathrm{E}_{A_{\mathfrak{m}_i}}(k_i)$ is a fixed injection.

Proof. The following diagram is easily seen to be commutative:

$$\begin{tikzcd}
\bigoplus_{i=1}^{s} W(k_i) \arrow[rr, "\sum F^{semi}_{\iota_i}"] \arrow[d, "\mathrm{diag}(F_{\iota_1},\dots,F_{\iota_s})"'] & & W^n(D^{(n)}_{A,\mu_I}, I_\bullet) \arrow[dll, "\simeq"', "\mathrm{loc}^n_A"] \\
\bigoplus_{i=1}^{s} \tilde{W}^n_{\mathfrak{m}_i A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}, (I_\bullet)_{\mathfrak{m}_i}) , & &
\end{tikzcd} \tag{10}$$

where loc^n_A is induced by localization. The homomorphism loc^n_A is an isomorphism by Theorem 5.2 and so the result follows from the considerations above and Theorem 3.2. $\square$

4. The length index

4.1. Recall the definition of the fundamental ideal $I(k)$ of the Witt group $W(k)$ of a field k. Since hyperbolic spaces have even dimension we have a well defined homomorphism:

$$e^0_k : W(k) \longrightarrow \mathbb{Z}/2\mathbb{Z} \qquad [V, \phi] \longmapsto \dim_k V + 2\mathbb{Z},$$

the *dimension index*. The kernel of this homomorphism is the fundamental ideal $I(k)$. We denote the powers of this ideal by $I^d(k)$ for $d \in \mathbb{N}$, and set $I^d(k) = W(k)$ for $d \leq 0$.

Let now (again) $(A, \mathfrak{m}, k)$ be a local ring with minimal dualizing complex $I_\bullet$, and $n = \mu_I(\mathfrak{m})$. We set further $\mathfrak{D}_I := \mathrm{Hom}_A(-, I_\bullet)$.

The natural extension of the function $\dim_k$ to $\mathcal{M}_{fl}(A)$ is the length function: $M \mapsto \operatorname{length} M$, whose extension to $D^b_{coh,\,\mathfrak{m}}(\mathcal{M}(A))$ is the *Euler characteristic*:

$$\chi : D^b_{coh,\,\mathfrak{m}}(\mathcal{M}(A)) \longrightarrow \mathbb{Z} \qquad K_\bullet \longmapsto \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{length} H_i(K_\bullet).$$

Lemma 4.2. *Let $K_\bullet$, $L_\bullet$, and $M_\bullet$ be complexes in $D^b_{coh,\,\mathfrak{m}}(\mathcal{M}(A))$.*

(i) *If* $K_\bullet \longrightarrow L_\bullet \longrightarrow M_\bullet \longrightarrow K_\bullet[1]$ *is an exact triangle then*

$$\chi(L_\bullet) = \chi(M_\bullet) + \chi(K_\bullet).$$

(ii) *We have* $\chi(M_\bullet) = -\chi(M_\bullet[1])$ *and* $\chi(M_\bullet) = (-1)^n \chi(\mathfrak{D}_I(M_\bullet))$.

Proof. (i) and the first assertion in (ii) are obvious. To prove the second assertion we can assume that $M_\bullet \in D^b(\mathcal{M}_{fl}(A))$, since the natural functor $D^b(\mathcal{M}_{fl}(A)) \longrightarrow D^b_{coh,\mathfrak{m}}(\mathcal{M}(A))$ is an equivalence. Denote by E the injective hull of the residue field k. Then we have an isomorphism $\mathrm{Hom}_A(M_\bullet, E)[n] \simeq \mathfrak{D}_I(M_\bullet)$ in $D^b(\mathcal{M}_{fl}(A))$ and so by the first assertion of (ii) it is enough to show $\chi(M_\bullet) = \chi(\mathrm{Hom}_A(M_\bullet, E))$. We prove this by induction on the cardinality of the set $\{i \in \mathbb{Z} | M_i \neq 0\}$. If $M_\bullet$ is concentrated in one degree, this is [8, Prop. 3.2.12 (b)]; otherwise there exists an exact triangle (e.g. a brutal truncation)

$$M'_\bullet \longrightarrow M''_\bullet \longrightarrow M_\bullet \longrightarrow M'_\bullet[1]$$

with $|\{i \in \mathbb{Z} \mid M'_i \neq 0\}|, |\{i \in \mathbb{Z} \mid M''_i \neq 0\}| < |\{i \in \mathbb{Z} \mid M_i \neq 0\}|$. By induction we know $\chi(M'_\bullet) = \chi(\mathrm{Hom}_A(M'_\bullet, E))$ and $\chi(M''_\bullet) = \chi(\mathrm{Hom}_A(M''_\bullet, E))$ and so the claim follows from part (i) of the lemma and the fact that $\mathrm{Hom}_A(-, E)$ is an exact functor on $\mathcal{M}_{fl}(A)$ (cf. [8, Prop. 3.2.12]). □

If an n-symmetric space $[M_\bullet, \varphi]$ is neutral, then there exists by definition an exact triangle

$$\mathfrak{D}_I(L_\bullet)[n-1] \longrightarrow L_\bullet \longrightarrow M_\bullet \longrightarrow \mathfrak{D}_I(L_\bullet)[n],$$

see [3, Sect. 2], and therefore $\chi(M_\bullet)$ is even. We get a well defined homomorphism

$$\ell_A : \tilde{W}^n_{\mathfrak{m}}(A, I_\bullet) \longrightarrow \mathbb{Z}/2\mathbb{Z} \qquad [M_\bullet, \varphi] \longmapsto \chi(M_\bullet).$$

Definition 4.3. The homomorphism

$$\ell_A : \tilde{W}^n_{\mathfrak{m}}(A, I_\bullet) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is called the *length index*.

Remark 4.4. The following diagram commutes:

$$\begin{tikzcd} W(k) \arrow[rr, "F_\iota"] \arrow[dr, "e^0_k"'] & & \tilde{W}^n_{\mathfrak{m}}(A, I_\bullet) \arrow[dl, "\ell_A"] \\ & \mathbb{Z}/2\mathbb{Z} & \end{tikzcd} \tag{11}$$

for all injective maps $\iota : k \hookrightarrow \mathrm{E}_A(k) = I_{-n}$. In particular, the image $F_\iota(I(k))$ does not depend on the choice of ι.

5. The Gersten–Witt spectral sequence

5.1. Let X be a scheme and $I_\bullet$ a minimal dualizing complex of X:

$$0 \longrightarrow I_{-m} \xrightarrow{d^I_{-m}} I_{-m-1} \xrightarrow{d^I_{-m-1}} \cdots \xrightarrow{d^I_{-n+2}} I_{-n+1} \xrightarrow{d^I_{-n+1}} I_{-n} \longrightarrow 0, \tag{12}$$

where $m = \min \mu_I$ and $n = \max \mu_I$. We set $X^{(p)}_{\mu_I} := \{x \in X \mid \mu_I(x) = p\}$ and define a finite filtration of $D^b_{coh}(\mathcal{M}(X))$:

$$D^b_{coh}(\mathcal{M}(X)) = D^{(m)}_{X,\mu_I} \supseteq D^{(m+1)}_{X,\mu_I} \supseteq \cdots \supseteq D^{(n-1)}_{X,\mu_I} \supseteq D^{(n)}_{X,\mu_I} \supseteq D^{(n+1)}_{X,\mu_I} = 0,$$

by setting

$$D^{(p)}_{X,\mu_I} := \{M_\bullet \in \mathcal{M}_{coh}(X) \mid \mu_I(x) \geq p \text{ for all } x \in \operatorname{supp} M_\bullet\}.$$

Since $D^{(p+1)}_{X,\mu_I}$ is a saturated subcategory of $D^{(p)}_{X,\mu_I}$ we have exact sequences of triangulated categories

$$D^{(p+1)}_{X,\mu_I} \rightarrowtail D^{(p)}_{X,\mu_I} \twoheadrightarrow D^{(p)}_{X,\mu_I}/D^{(p+1)}_{X,\mu_I}. \tag{13}$$

for all $m \le p \le n$. It is easy to see that $\mathfrak{D}_I(D^{(p)}_{X,\mu_I}) \subseteq D^{(p)}_{X,\mu_I}$, and so this subcategory is a triangulated category with duality, too. In particular, (13) is an exact sequence of triangulated categories with duality and so we get from the localization theorem of Balmer [3] a long exact sequence

$$\longrightarrow W^i(D^{(p)}_{X,\mu_I}, I_\bullet) \longrightarrow W^i(D^{(p)}_{X,\mu_I}/D^{(p+1)}_{X,\mu_I}, I_\bullet) \xrightarrow{\partial} W^{i+1}(D^{(p+1)}_{X,\mu_I}, I_\bullet) \longrightarrow$$

for all $n \ge p \ge m$. The collection of these sequences is an exact couple giving us a convergent spectral sequence (cf. [20, Chap. XI.5]):

$$E_1^{p,q}(X, I_\bullet) := W^{p+q}(D^{(p)}_{X,\mu_I}/D^{(p+1)}_{X,\mu_I}, I_\bullet) \Longrightarrow \tilde{W}^{p+q}(X, I_\bullet),$$

the *Gersten–Witt spectral sequence* of X with respect to $I_\bullet$. The 0-th line of this spectral sequence is the (natural) *Gersten–Witt complex* $\mathrm{GW}(X, I_\bullet)$.

If $K_\bullet \in D^{(p)}_{X,\mu_I}$ and $x \in X^{(p)}_{\mu_I}$ then the homology groups of $(K_\bullet)_x$ are finite length $\mathcal{O}_{X,x}$-modules. Indeed for any $x \neq y \in \operatorname{Spec} \mathcal{O}_{X,x} \subseteq X$ we have $\mu_I(y) \le p-1$ by Lemma 1.14, and therefore $(K_\bullet)_y = 0$. It follows that localization induces a functor

$$\Gamma^p_{X,\mu_I} : D^{(p)}_{X,\mu_I}/D^{(p+1)}_{X,\mu_I} \longrightarrow \coprod_{\mu_I(x)=p} D^b_{coh,\,\mathfrak{m}_x}(\mathcal{M}(\mathcal{O}_{X,x})),$$

where $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{X,x}$. The following is well known.

Theorem 5.2. *The functor Γ^p_{X,μ_I} is an equivalence.*

Proof. We set $D^p := D^{(p)}_{X,\mu_I}$ and

$$\mathcal{M}^p := \{M \in \mathcal{M}_{coh}(X) \mid \mu_I(x) \ge p \text{ for all } x \in \operatorname{supp} M\}.$$

Note that $\mathcal{M}^{p+1}$ is a Serre subcategory of $\mathcal{M}^p$ and so the localization $\mathcal{M}^p/\mathcal{M}^{p+1}$ exists. We refer the reader to [9, Chap. 14] for facts about localizations of abelian categories.

Since X is noetherian the natural functors

$$D^b(\mathcal{M}^p) \longrightarrow D^p \qquad \text{and} \qquad D^b(\mathcal{M}_{fl}(\mathcal{O}_{X,x})) \longrightarrow D^b_{coh,\,\mathfrak{m}_x}(\mathcal{M}(\mathcal{O}_{X,x}))$$

are equivalences, see e.g. [18, 1.15], and therefore it is enough to prove that the localization functor induces an equivalence

$$D^b(\mathcal{M}^p/\mathcal{M}^{p+1}) \xrightarrow{\simeq} \prod_{\mu_I(x)=p} D^b(\mathcal{M}_{fl}(\mathcal{O}_{X,x})),$$

since by a result of Grothendieck, see e.g. [18, Lem. 1.15], we have an exact sequence

$$D^b(\mathcal{M}^{p+1}) \rightarrowtail D^b(\mathcal{M}^p) \twoheadrightarrow D^b(\mathcal{M}^p/\mathcal{M}^{p+1})$$

and hence an equivalence

$$D^b(\mathcal{M}^p)/D^b(\mathcal{M}^{p+1}) \simeq D^b(\mathcal{M}^p/\mathcal{M}^{p+1}).$$

Hence the theorem follows from the following result which is due to Gabriel [11]. $\square$

Theorem 5.3. *The localization functor induces an equivalence*

$$\gamma^p := \gamma^p_{X,\mu_I} : \mathcal{M}^p/\mathcal{M}^{p+1} \xrightarrow{\simeq} \prod_{\mu_I(x)=p} \mathcal{M}_{fl}(\mathcal{O}_{X,x}).$$

Proof. We first show that γ^p is surjective (up to isomorphism) on objects. Let $x \in X^{(p)}_{\mu_I}$, and $M \in \mathcal{M}_{fl}(\mathcal{O}_{X,x})$. Since M has finite length there exists an affine neighborhood $\kappa : \operatorname{Spec} A \subseteq X$ of x and an exact sequence of A-modules

$$(A/P^m)^{b_1} \xrightarrow{\alpha} (A/P^m)^{b_2} \longrightarrow M' \longrightarrow 0,$$

where $P \in \operatorname{Spec} A$ corresponds to x and b_1, b_2, m are positive integers such that $M'_P \simeq M$. Obviously $\operatorname{supp} M' \subseteq \operatorname{Spec} A/P$. By [16, Cor. 6.9.3] there exists a coherent $\mathcal{O}_X$-module N such that $N|_{\operatorname{Spec} A} \simeq M'$ and $N \subseteq \kappa_*(M')$. In particular, $N_x \simeq M$ and $N_y = 0$ if $y \notin \overline{\{x\}}$. It follows that $N \in \mathcal{M}^p$ and $N_y = 0$ for all $y \in X^{(p)}_{\mu_I}$. Therefore γ^p is surjective.

That γ^p is faithful is obvious. That it is full, too, is a straightforward consequence of the next two lemmas. □

Lemma 5.4. *Let $x \in X^{(p)}_{\mu_I}$, $M, N \in \mathcal{M}^p$ with $M_y = 0$ for all $y \in X^{(p)}_{\mu_I} \setminus \{x\}$, and $f : M_x \longrightarrow N_x$ a homomorphism in $\mathcal{M}_{fl}(\mathcal{O}_{X,x})$. Then there exists a morphism $f' : M \longrightarrow N$ in $\mathcal{M}^p/\mathcal{M}^{p+1}$ such that $f'_x = f$.*

Proof. Since localization commutes with the Hom-functor for finitely generated modules we can extend f to an affine neighborhood $U = \operatorname{Spec} A$ of x. Let Z be the complement of $\operatorname{Spec} A$ in X and $\mathcal{J}$ the ideal defining Z. By [16, Prop. 6.9.17] this extension is in the image of the natural homomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{J}^m M, N) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(M|_U, N|_U)$$

for some $m \geq 0$. We are done, since $M/\mathcal{J}^m M \in \mathcal{M}^{p+1}$. □

Lemma 5.5. *Let $M \in \mathcal{M}^p$. Then there exists for all $x \in X^{(p)}_{\mu_I}$ a coherent $\mathcal{O}_X$-module $M(x)$ and a morphism $c(x) : M(x) \longrightarrow M$ in $\mathcal{M}^p/\mathcal{M}^{p+1}$ such that*

(i) *$M(x)_y = 0$ if $y \in X^{(p)}_{\mu_I} \setminus \{x\}$, and*
(ii) *$c(x)_x : M(x)_x \longrightarrow M_x$ is an isomorphism.*

In particular, the morphism

$$\bigoplus_{x \in X^{(p)}_{\mu_I}} M(x) \xrightarrow{\sum c(x)} M$$

is an isomorphism in $\mathcal{M}^p/\mathcal{M}^{p+1}$.

Proof. We choose an affine neighborhood $U = \operatorname{Spec} A$ of $x \in X^{(p)}_{\mu_I}$, and denote by P the corresponding prime ideal of A. We set

$$M(P) := \{m \in \Gamma(U, M) \mid P^l \cdot m = 0 \text{ for some } l \geq 0\}.$$

This is a finitely generated A-module, and we have $M(P)_P = M_x$ and $M(P)_Q = 0$ for all $Q \in \operatorname{Spec} A \cap X^{(p)}_{\mu_I}$. As shown above, there exists a coherent $\mathcal{O}_X$-module $M(x)$ with $M(x)_x = M(P)_P$ and $M(x)_y = 0$ for all $y \neq x$ with $\mu_I(y) = p$. By Lemma 5.4 the isomorphism (in fact the identity) $M(x)_x \longrightarrow M_x$ can be extended to a morphism $c(x) : M(x) \longrightarrow M$ in $\mathcal{M}^p/\mathcal{M}^{p+1}$. We are done. □

5.6. We get an isomorphism

$$\operatorname{loc}^p_X = \operatorname{loc}^p_{X,I} \ : \ W^p(D^{(p)}_{X,\mu_I}/D^{(p+1)}_{X,\mu_I}, I_\bullet) \xrightarrow{\simeq} \bigoplus_{\mu_I(x)=p} \tilde{W}^p_{\mathfrak{m}_x}(\mathcal{O}_{X,x}, (I_\bullet)_x)$$

for all $p \in \mathbb{Z}$. In particular, by Theorem 3.2 we have $E_1^{p,q}(X, I_\bullet) = 0$ if $q \not\equiv 0 \bmod 4$.

From Proposition 1.10 we get that $(I_\bullet)_x$ is a minimal dualizing complex because $I_\bullet$ is minimal, and so in particular, $(I_{-\mu_I(x)})_x = \mathrm{E}_{\mathcal{O}_{X,x}}(k(x))$ is an $\mathcal{O}_{X,x}$-injective hull of the residue field $k(x)$. We choose an injection $\iota_x : k(x) \hookrightarrow \mathrm{E}_{\mathcal{O}_{X,x}}(k(x))$ for all $x \in X$.

Notation 5.7. We call $\underline{\iota} := (\iota_x)_{x \in X}$ a *family of injective hull injections* for X.

5.8. Associated with such a family $\underline{\iota}$ we have by Theorem 3.2 above an isomorphism

$$(F_{\iota_x})_{x\in X^{(p)}_{\mu_I}}\;:\;\bigoplus_{\mu_I(x)=p} W(k(x)) \xrightarrow{\simeq} \bigoplus_{\mu_I(x)=p} \tilde{W}^p_{\mathfrak{m}_x}(\mathcal{O}_{X,x},(I_\bullet)_x)$$

for all $p\in\mathbb{Z}$. We define d^p_X and $d^p_{\underline{\iota}}$ as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & W^p(D^{(p)}_{X,\mu_I}/D^{(p+1)}_{X,\mu_I}, I_\bullet) \arrow[r, "\partial^p_X"] \arrow[d, "\mathrm{loc}^p_X"', "\simeq"] & W^{p+1}(D^{(p+1)}_{X,\mu_I}/D^{(p+2)}_{X,\mu_I}, I_\bullet) \arrow[r] \arrow[d, "\simeq"', "\mathrm{loc}^{p+1}_X"] & \cdots \\
\cdots \arrow[r] & \bigoplus_{\mu_I(x)=p} \tilde{W}^p_{\mathfrak{m}_x}(\mathcal{O}_{X,x},(I_\bullet)_x) \arrow[r, "d^p_X"] & \bigoplus_{\mu_I(x)=p+1} \tilde{W}^{p+1}_{\mathfrak{m}_x}(\mathcal{O}_{X,x},(I_\bullet)_x) \arrow[r] & \cdots \\
\cdots \arrow[r] & \bigoplus_{\mu_I(x)=p} W(k(x)) \arrow[r, "d^p_{\underline{\iota}}"] \arrow[u, "(F_{\iota_x})_{x\in X^{(p)}_{\mu_I}}", "\simeq"'] & \bigoplus_{\mu_I(x)=p+1} W(k(x)) \arrow[r] \arrow[u, "\simeq", "(F_{\iota_x})_{x\in X^{(p+1)}_{\mu_I}}"'] & \cdots .
\end{tikzcd} \tag{14}$$

The bottom line of this diagram is called the *Gersten–Witt complex* associated with the family of injective hull injections $\underline{\iota}$. We denote this complex by $\mathrm{GW}(X, I_\bullet, \underline{\iota})$ in the following.

We list some functorial properties of this complex.

5.9. Let $s\in\mathbb{Z}$; then $I_\bullet[s]$ is a dualizing complex of X, too, and $D^{(p)}_{X,\mu_{I[s]}} = D^{(p+s)}_{X,\mu_I}$. Therefore the shift functor (9) induces an isomorphism of spectral sequences

$$(\mathrm{id}_{D^b_{coh}(\mathcal{M}(X))}, \mathfrak{sh}^s)_*\;:\;E_1^{p,q}(X, I_\bullet[s]) \longrightarrow E_1^{p+s,q}(X, I_\bullet),$$

and so by Remark 3.3 an isomorphism of Gersten–Witt complexes:

$$\begin{tikzcd}
\mathrm{GW}(X, I_\bullet[s], \underline{\iota})\;: & \bigoplus_{\mu_{I[s]}(x)=p} W(k(x)) \arrow[r] \arrow[d, "\mathrm{id}"] & \bigoplus_{\mu_{I[s]}(x)=p+1} W(k(x)) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\mathrm{GW}(X, I_\bullet, \underline{\iota})\;: & \bigoplus_{\mu_I(x)=p+s} W(k(x)) \arrow[r] & \bigoplus_{\mu_I(x)=p+s+1} W(k(x)) \arrow[r] & \cdots .
\end{tikzcd}$$

5.10. If κ is an open immersion $U\subseteq X$ or an inclusion $U=\mathrm{Spec}\,\mathcal{O}_{X,x}\hookrightarrow X$, then $\underline{\iota}|_U = (\iota_x)_{x\in U}$ is a family of injective hull injections for U, and $\kappa^*(I_\bullet)$ a minimal dualizing complex of U by Proposition 1.10. We have $\kappa^*(D^{(p)}_{X,\mu_I})\subseteq D^{(p)}_{U,\mu_{\kappa^*(I)}}$ and therefore a morphism of spectral sequences

$$\kappa^*\;:\;E^{p,q}(X, I_\bullet) \longrightarrow E^{p,q}(U, \kappa^*(I_\bullet)).$$

Obviously this morphism induces a morphism of complexes

$$\kappa^*\;:\;\mathrm{GW}(X, I_\bullet, \underline{\iota}) \longrightarrow \mathrm{GW}(U, \kappa^*(I_\bullet), \underline{\iota}|_U)$$

which is the natural projection in every degree.

5.11. Assume now that $X=\mathrm{Spec}\,A$ is affine, and $J\subseteq A$ an ideal. Let $\pi: A\longrightarrow A/J$ be the projection. Then $\pi^\natural(I_\bullet)$ is a dualizing complex of A/J by Proposition 1.10.

The restriction of scalars functor $\pi_* : D^b_{coh}(\mathcal{M}(A/J)) \longrightarrow D^b_{coh}(\mathcal{M}(A))$ respects the filtration induced by the codimension functions: $\pi_*(D^{(p)}_{A/J,\mu_{\pi^\natural(I)}}) \subseteq D^{(p)}_{A,\mu_I}$, and so the transfer morphism $\mathrm{Tr}_{(A/J)/A}$ of [13] induces a morphism of spectral sequences:

$$\mathrm{Tr}_{(A/J)/A} : E^{p,q}(A/J, \pi^\natural(I_\bullet)) \longrightarrow E^{p,q}(A, I_\bullet);$$

see [12, Thm. 2.9]. The transfer commutes with localization, see [13, Sect. 4], and therefore the diagram

$$\begin{tikzcd}
W^p(D^{(p)}_{A/J,\mu_{\pi^\natural(I)}}/D^{(p+1)}_{A/J,\mu_{\pi^\natural(I)}}, \pi^\natural(I_\bullet)) \arrow[r, "\mathrm{loc}^p_{A/J}"] \arrow[d, "\mathrm{Tr}_{(A/J)/A}"'] & \bigoplus_{\mu_{\pi^\natural(I)}(P/J)=p} \tilde{W}^p_{P(A/J)_P}((A/J)_P, \pi^\natural(I_\bullet)_P) \arrow[d, "(\mathrm{Tr}_{(A/J)_P/A_P})_{\mu_{\pi^\natural(I)}(P/J)=p}"'] \\
W^p(D^{(p)}_{A,\mu_I}/D^{(p+1)}_{A,\mu_I}, I_\bullet) \arrow[r, "\mathrm{loc}^p_A"'] & \bigoplus_{\mu_I(P)=p} \tilde{W}^p_{PA_P}(A_P, (I_\bullet)_P)
\end{tikzcd}$$

commutes for all $p \in \mathbb{Z}$. By Proposition 3.6 we can choose for any $P/J \in \operatorname{Spec} A/J$ with $\mu_{\pi^\natural(I)}(P/J) = p$ an injective hull injection $\iota_{P/J} : k(P) \hookrightarrow \mathrm{E}_{(A/J)_P}(k(P))$, where $k(P) = A_P/PA_P$ denotes the residue field of the prime ideal P/J of A/J such that the diagram

$$\begin{tikzcd}
W(k(P)) \arrow[r, "F_{\iota_{P/J}}"] \arrow[d, "="'] & \tilde{W}^p_{P(A/J)_P}((A/J)_P, \pi^\natural(I_\bullet)_P) \arrow[d, "\mathrm{Tr}_{(A/J)_P/A_P}"] \\
W(k(P)) \arrow[r, "F_{\iota_P}"] & \tilde{W}^p_{PA_P}(A_P, (I_\bullet)_P)
\end{tikzcd}$$

commutes. Hence there exists a family of injective hull injections $\underline{\iota}_J$ for A/J such that $\mathrm{Tr}_{(A/J)/A}$ induces a morphism of Gersten–Witt complexes

$$\mathrm{GW}(A/J, \pi^\natural(I_\bullet), \underline{\iota}_J) \longrightarrow \mathrm{GW}(X, I_\bullet, \underline{\iota})$$

which is the natural injection $\bigoplus_{\mu_{\pi^\natural(I)}(P/J)=p} W(k(P)) \longrightarrow \bigoplus_{\mu_I(P)=p} W(k(P))$ in every degree.

Remark 5.12. Let $\pi : A \longrightarrow A/J$ be as above. The transfer induces a homomorphism $\mathrm{Tr}_{(A/J)/A} : \tilde{W}^i(A/J, \pi^\natural(I_\bullet)) \longrightarrow \tilde{W}^i_J(A, I_\bullet)$ for all $i \in \mathbb{Z}$, which is an isomorphism. This can be proven as in the Gorenstein case using the Gersten–Witt spectral sequence; cf. [12, Sect. 4].

6. Generalized second residue homomorphisms

6.1. Let $(A, \mathfrak{m}, k)$ be a one-dimensional local domain with dualizing complex. Since A is Cohen–Macaulay it has then a dualizing module Ω; see [13, Thm. 6.2]. A minimal injective resolution $I_0 \xrightarrow{d^I_0} I_{-1}$ of Ω is a dualizing complex of A; cf. Example 1.8. Choosing a family of injective hull injections $\underline{\iota} = (\iota_{(0)}, \iota_{\mathfrak{m}})$ for A we get from (14) a commutative diagram:

$$\begin{tikzcd}
W^0(D^{(0)}_{A,\mu_I}/D^{(1)}_{A,\mu_I}, I_\bullet) \arrow[r, "\partial^0_A"] \arrow[d, "\mathrm{loc}^0_A", "\simeq"'] & W^1(D^{(1)}_{A,\mu_I}, I_\bullet) \arrow[d, "="] \\
\tilde{W}^0(K, (I_\bullet)_{(0)}) \arrow[r, "d^0_A"] & \tilde{W}^1_{\mathfrak{m}}(A, I_\bullet) \\
W(K) \arrow[r, "d^0_{\underline{\iota}}"] \arrow[u, "F_{\iota_{(0)}}", "\simeq"'] & W(k). \arrow[u, "F_{\iota_{\mathfrak{m}}}"', "\simeq"]
\end{tikzcd} \tag{15}$$

Definition 6.2. We call the homomorphism $d^0_{\underline{\iota}}$ the *generalized second residue homomorphism* corresponding to $\underline{\iota}$.

Lemma 6.3. *The generalized second residue homomorphism is $W(A)$-linear.*

Proof. This is clear since all the maps involved are $W(A)$-linear. In fact, $F_{\iota_{(0)}}$ and $F_{\iota_{\mathfrak{m}}}$ are $W(A)$-linear, see Theorem 3.2, and the differential ∂^0_A of the Gersten–Witt complex by [15, Thm. 2.10]. □

6.4. Assume A is regular, i.e. a discrete valuation ring, and π an uniformizer, i.e. a generator of the maximal ideal $\mathfrak{m}$. For any such uniformizer we have a second residue homomorphism $\delta^\pi : W(K) \longrightarrow W(k)$. It is uniquely determined by the following property. Any symmetric space α over K has a decomposition

$$\alpha \simeq <a_1, \ldots, a_r> \perp \pi <a_{r+1}, \ldots, a_{r+s}>$$

with $a_1, \ldots, a_{r+s} \in A^\times = A \setminus \mathfrak{m}$. Then $\delta^\pi(\alpha) = <\bar{a}_{r+1}, \ldots, \bar{a}_{r+s}>$, where $\bar{x}$ denotes the residue class of x in k.

Proposition 6.5. *If A is regular, then there exists an uniformizer $\pi \in \mathfrak{m}$ such that $d^0_{\underline{\iota}} = \delta^\pi$.*

Proof. Obviously it is enough to show that there exists an uniformizer π such that $\delta^\pi(<x>) = d^0_{\underline{\iota}}(<x>)$ for all $0 \neq x \in K$. To show this let π' be any uniformizer of A. Then we have $x = a \cdot \pi'^s$ for some $s \in \mathbb{Z}$ and $a \in A^\times$. If s is even we have $<x> = <a>$ in $W(K)$ and therefore $\delta^\pi(<x>) = 0$ for all uniformizers π. But $d^0_{\underline{\iota}}(<x>) = d^0_{\underline{\iota}}(<a>) = 0$, too, because the cone of the isomorphism $A \xrightarrow{\cdot a} A$ has zero homology. If s is odd then $<x> = <a \cdot \pi'>$ in $W(K)$, and so $\delta^\pi(<x>) = <\bar{a}> \otimes <\overline{\left(\frac{\pi'}{\pi}\right)}>$ for all uniformizers π. We have $d^0_{\underline{\iota}}(<x>) = <\bar{a}> \otimes d^0_{\underline{\iota}}(<\pi'>)$ since by Lemma 6.3 above the morphism $d^0_{\underline{\iota}}$ is $W(A)$-linear. Now the cone of the homomorphism $A \xrightarrow{\cdot \pi'} A$ is isomorphic to $k = A/A\pi'$, and so $d^0_{\underline{\iota}}(<\pi'>) = <\bar{b}>$ for some $b \in A^\times$, since the natural functor $\mathcal{M}_{fl}(A) \longrightarrow D^b(\mathcal{M}_{fl}(A))$ is full and faithful. Choosing $\pi = b^{-1}\pi'$ we get the result. □

Theorem 6.6. *Let $(A, \mathfrak{m}, k)$ be a one-dimensional local domain (which is automatically Cohen–Macaulay) with dualizing module Ω. Then for any family of injective hull injections $\underline{\iota} = (\iota_{(0)}, \iota_{\mathfrak{m}})$ and all $d \in \mathbb{Z}$ we have*

$$d^0_{\underline{\iota}}(I^d(K)) \subseteq I^{d-1}(k),$$

where K is the quotient field of A.

6.7. This has been proven by Arason [1] if the ring A is regular, i.e. a discrete valuation ring. In the case where A is essentially of finite type over a field one can reduce to this using normalization. But this is impossible in our more general setting where the normalization morphism need not be finite.

We use Pfister forms in our proof, as well as the classical transfer morphism (also called Scharlau transfer by some authors) for Witt groups of fields. For the convenience of our reader we recall briefly the necessary facts referring to [30] for details and more information. A d-Pfister form over K is a diagonal form

$$\ll a_1, \ldots, a_d \gg := \bigotimes_{i=1}^{d} <1, -a_i>$$

where $a_i \in K^\times$ for all $1 \leq i \leq d$. The d-th power of the fundamental ideal is generated by d-Pfister forms, because we have $<a, b> = <1, a> - <1, -b>$ in $W(K)$. Let now L/K be a finite field extension and $h : L \longrightarrow K$ a K-linear map which is not zero. Then the mapping $[V, \psi] \longmapsto [V, h \cdot \psi]$, where we consider V as a K-vector space on the right hand side, defines a homomorphism $\mathrm{cor}^h_{L/K} : W(L) \longrightarrow W(K)$. This homomorphism depends on the K-linear map h and will be called a classical transfer (or a trace map).

The following is crucial for the proof of Theorem 6.6.

Lemma 6.8. *Let $\ll a, b \gg$ be a 2-Pfister form over the quotient field K of the one-dimensional local domain A. Then there exists a finite extension $K \supset B \supset A$ of A, and a unit $c \in B^\times$ such that*

$$\ll a, b \gg = \ll c, d \gg \quad \text{in } W(K)$$

for some $d \in K^\times$. The ring B is semilocal and has Krull dimension 1.

Proof. The last statement follows from the Krull–Akizuki Theorem [22, Thm. 11.7]. From this result we also deduce that the integral closure $\tilde{A}$ of A in K is a semilocal Dedekind domain and hence a principal ideal domain. Let $\{\pi_1, \ldots, \pi_s\} \subset \tilde{A}$ be the finite set of (up to units) all irreducible elements and denote $v_1, \ldots, v_s$ the corresponding valuations of K. Since for any unit $c \in \tilde{A}^\times$ the ring $A[c, c^{-1}]$ is a finite extension of A it is enough to show that there exists $c \in \tilde{A}^\times$ with $\langle\langle a, b\rangle\rangle = \langle\langle c, d\rangle\rangle$ in $W(K)$ for some $d \in K^\times$.

Since $<xy^2> = <x>$ in $W(K)$ for all $x, y \in K^\times$ we can assume that $a = \alpha \cdot \prod_{i=1}^s \pi_i^{\gamma_i}$ and $b = \beta \cdot \prod_{i=1}^s \pi_i^{\epsilon_i}$ with $\alpha, \beta \in \tilde{A}^\times$ and $\gamma_i, \epsilon_i \in \{0; 1\}$ for all $1 \le i \le s$. We define now integers ρ_i, ϱ_i and σ_i as follows. If $(\gamma_i, \epsilon_i) = (0, 0)$ we set $(\rho_i, \varrho_i, \sigma_i) := (1, 1, 0)$, if $(\gamma_i, \epsilon_i) = (0, 1)$ or $= (1, 0)$ we set $(\rho_i, \varrho_i, \sigma_i) := (0, 0, 0)$, and if $(\gamma_i, \epsilon_i) = (1, 1)$ we set $(\rho_i, \varrho_i, \sigma_i) := (0, 0, -1)$. Let now

$$x = \prod_{i=1}^s \pi_i^{\rho_i}, \quad y = \prod_{i=1}^s \pi_i^{\varrho_i} \quad \text{and} \quad z = \prod_{i=1}^s \pi_i^{\sigma_i} \quad \in K^\times.$$

Then $c := -ax^2 - by^2 + abz^2$ is a unit in $\tilde{A}$, because $v_i(c) = 0$ for all $1 \le i \le s$ as our reader will easily verify. But this means that the pure subform $< -a, -b, ab>$ of $\langle\langle a, b\rangle\rangle$ represents $c \in \tilde{A}^\times$ and so $\langle\langle a, b\rangle\rangle = \langle\langle c, d\rangle\rangle$ for some $d \in K^\times$ by [30, Chap. 4, Thm. 1.4]. $\square$

6.9. *Proof of* Theorem 6.6. The statement is obvious for $d \le 1$, so let $d \ge 2$. Since $I^d(K)$ is generated by d-Pfister forms it is enough to show that $d^0_{\underline{\iota}}(\alpha) \in I^{d-1}(k)$ for all Pfister forms $\alpha = \langle\langle a_1, \ldots, a_d\rangle\rangle$. By Lemma 6.8 there exists a finite semilocal extension $K \supset B$ of A, a unit $c \in B^\times$, and a $(d-1)$-Pfister form β such that $\alpha = \langle\langle c\rangle\rangle \otimes \beta$. Let $\{\mathfrak{n}_1, \ldots, \mathfrak{n}_s\} \subset \operatorname{Spec} B$ be the finite set of maximal ideals of B and denote as $l_i = B/\mathfrak{n}_i$ their respective residue fields. By Proposition 6.10 below there exist families of injective hull injections $\underline{j}_i$ for $B_{\mathfrak{n}_i}$ and non-zero k-linear maps $h_i : l_i \longrightarrow k$ such that $d^0_{\underline{\iota}} = \sum_{i=1}^s \operatorname{cor}^{h_i}_{l_i/k} d^0_{\underline{j}_i}$. Therefore we have

$$d^0_{\underline{\iota}}(\alpha) = \sum_{i=1}^s \operatorname{cor}^{h_i}_{l_i/k} d^0_{\underline{j}_i}(\langle\langle c\rangle\rangle \otimes \beta) = \sum_{i=1}^s \operatorname{cor}^{h_i}_{l_i/k} \Big(\langle\langle c\rangle\rangle \otimes d^0_{\underline{j}_i}(\beta)\Big)$$

(the latter by Lemma 6.3). By induction we can assume that $d^0_{\underline{j}_i}(\beta) \in I^{d-2}(l_i)$ for all $1 \le i \le s$. But by a result of Arason [1, Satz 3.3] we know that $\operatorname{cor}^{h_i}_{l_i/k}(I^{d-1}(l_i)) \subseteq I^{d-1}(k)$, and the theorem follows. $\square$

Proposition 6.10. *Let $(A, \mathfrak{m}, k)$ be a local one-dimensional domain with dualizing module Ω, quotient field K, and a family of injective hull injections $\underline{\iota}$. Let further $L \supseteq K$ be a finite field extension and $L \supset B \supseteq A$ be a finite A-module (which is by the Krull–Akizuki Theorem a semilocal one-dimensional domain) such that the quotient field of B is L. Denote by $\mathfrak{n}_1, \ldots, \mathfrak{n}_s$ the finitely many maximal ideals of B and $l_i = B/\mathfrak{n}_i$ their respective residue fields. Then there is*

(i) *a non-zero K-linear map $h : L \longrightarrow K$, which can be chosen to be the identity if $K = L$,*
(ii) *a family of injective hull injections $\underline{j}_i$ for $B_{\mathfrak{n}_i}$, and*
(iii) *a non-zero k-linear map $h_i : l_i \longrightarrow k$*

for $i = 1, \ldots, s$ such that the following diagram commutes:

$$\begin{tikzcd} W(L) \arrow[r, "(d^0_{\underline{j}_i})_{i=1}^s"] \arrow[d, "\operatorname{cor}^h_{L/K}"'] & \bigoplus_{i=1}^s W(l_i) \arrow[d, "\sum_{i=1}^s \operatorname{cor}^{h_i}_{l_i/k}"] \\ W(K) \arrow[r, "d^0_{\underline{\iota}}"] & W(k). \end{tikzcd}$$

Proof. The B-module $\Omega' := \operatorname{Hom}_A(B, \Omega)$ is a dualizing module of B by [13, Cor. 6.3]. We have in this case a transfer morphism $\operatorname{Tr}_{B/A}$; see [13]. Recall its definition. Let $\Omega \longrightarrow I_0 \longrightarrow I_{-1}$ be a minimal injective resolution of Ω as above. The complex $I'_\bullet := \operatorname{Hom}_A(B, I_\bullet) \in D^b_{coh}(\mathcal{M}(B))$ is then an injective resolution of Ω' and we have an A-morphism $\zeta : I'_\bullet \longrightarrow I_\bullet$ given by

$$I'_r = \operatorname{Hom}_A(B, I_r) \ni f \longmapsto (-1)^r f(1) \in I_r$$

in degree $r \in \{-1; 0\}$. The induced morphism $\mathrm{Hom}_B(M_\bullet, I'_\bullet) \longrightarrow \mathrm{Hom}_A(M_\bullet, I_\bullet)$ is an isomorphism for all $M_\bullet \in D^b_{coh}(\mathcal{M}(B))$, and makes the restriction of scalars functor $D^b_{coh}(\mathcal{M}(B)) \longrightarrow D^b_{coh}(\mathcal{M}(A))$ duality preserving. Note that μ_I and $\mu_{I'}$ are the usual codimensions of A and B, respectively.

Sublemma 6.11. *The complex $I'_\bullet$ is a minimal injective resolution of Ω'.*

Proof. By Theorem 1.4 the injective B-module $\mathrm{Hom}_A(B, I_0) = \mathrm{Hom}_A(B, \mathrm{E}_A(A))$ is a direct sum of the indecomposable injective modules $\mathrm{E}_B(B), \mathrm{E}_B(l_1), \ldots, \mathrm{E}_B(l_s)$, and by Theorem 1.15 we have $I_0 = \mathrm{E}_A(A)$ and $I_{-1} = \mathrm{E}_A(k)$. Let $0 \neq a \in A$. Then $\mathrm{E}_A(A) \xrightarrow{\cdot a} \mathrm{E}_A(A)$ is an isomorphism and so for all $a \in A \setminus \{0\}$ and $0 \neq f \in \mathrm{Hom}_A(B, I_0)$ we have $a \cdot f \neq 0$. In particular, $\mathrm{E}_B(l_i)$ cannot appear in the decomposition of $\mathrm{Hom}_A(B, \mathrm{E}_A(A))$ into indecomposable injective modules for all $i = 1, \ldots, s$. We claim now that

$$I'_{-1} = \mathrm{Hom}_A(B, I_{-1}) = \bigoplus_{i=1}^{s} \mathrm{E}_B(l_i). \tag{16}$$

In fact any element of $I_{-1} = \mathrm{E}_A(k)$ is annihilated by some power of $\mathfrak{m}$, and hence any element of $\mathrm{Hom}_A(B, I_{-1})$, too, because B is a finite A-module. Therefore $\mathrm{E}_B(B)$ cannot appear in the decomposition of I'_{-1} into indecomposable injective modules. Since the natural homomorphism of complexes $l_i \longrightarrow \mathrm{Hom}_B(\mathrm{Hom}_B(l_i, I'_\bullet), I'_\bullet)$ (considering l_i as a complex which is concentrated in degree 0) is a quasi-isomorphism and $\mathrm{Hom}_B(l_i, \mathrm{E}_B(l_j)) = 0$ for $i \neq j$ by [12, Lem. 3.3] we get the claim. Therefore $(I_0)'_{(0)} \simeq L$ and so $I'_0 = \mathrm{E}_B(B)$. From this the sublemma follows by comparing the injective resolution $I'_\bullet$ with a minimal injective resolution of Ω'; cf. [12, Proof of Thm. 3.5]. $\square$

Sublemma 6.12. *There exist*

(i) *non-zero k-linear maps $h_i : l_i \longrightarrow k$, and*
(ii) *embeddings $j_{\mathfrak{n}_i} : l_i \hookrightarrow \mathrm{E}_B(l_i)$*

such that the following diagram commutes:

$$\begin{tikzcd} W^1(D^{(1)}_{B,\mu_{I'}}, I'_\bullet) \arrow[d, "\mathrm{Tr}_{B/A}"'] & W(l_i) \arrow[l, "F^{semi}_{j_{\mathfrak{n}_i}}"'] \arrow[d, "\mathrm{cor}^{h_i}_{l_i/k}"] \\ W^1(D^{(1)}_{A,\mu_I}, I_\bullet) & W(k) \arrow[l, "F_{\iota_{\mathfrak{m}}}"'] \end{tikzcd} \tag{17}$$

for all $i = 1, \ldots, s$. In particular, we have

$$(F_{\iota_{\mathfrak{m}}})^{-1} \cdot \mathrm{Tr}_{B/A} = \left(\sum_{i=1}^{s} \mathrm{cor}^{h_i}_{l_i/k} \right) \cdot \left(\sum_{i=1}^{s} F^{semi}_{j_{\mathfrak{n}_i}} \right)^{-1}$$

(see Section 3 for the definition of $F^{semi}_{j_{\mathfrak{n}_i}}$).

Proof. We consider for all $1 \leq i \leq s$ the diagram

$$\begin{tikzcd} l_i \arrow[r, "\exists\, j_{\mathfrak{n}_i}"] \arrow[drr, "\exists\, \tilde{h}_i"] & \mathrm{E}_B(l_i) \subseteq \mathrm{Hom}_A(B, \mathrm{E}_A(k)) \arrow[dr, "\zeta_{-1}"] & \\ k \arrow[u] \arrow[rr, "\iota_{\mathfrak{m}}"] & & \mathrm{E}_A(k), \end{tikzcd} \tag{18}$$

where the map $k \longrightarrow l_i$ is induced by the embedding $A \subseteq B$. Because $\mathrm{E}_A(k)$ is A-injective there exists an A-linear morphism $\tilde{h}_i$ such that the triangle lower left commutes. We define

$$j_{\mathfrak{n}_i} : l_i \longrightarrow \mathrm{Hom}_A(B, \mathrm{E}_A(k)) \qquad x \longmapsto \{b \mapsto -\tilde{h}_i(b \cdot x)\}.$$

This is obviously a B-linear morphism and injective because $\tilde{h}_i(1) = \iota_{\mathfrak{m}}(1) \neq 0$. From (16) we conclude that $\operatorname{Im} j_{\mathfrak{n}_i} \subseteq \mathrm{E}_B(l_i) \subseteq \operatorname{Hom}_A(B, I_{-1})$. Moreover, $j_{\mathfrak{n}_i}$ makes diagram (18) commutative. Since $\tilde{h}_i \neq 0$ we have

$$\operatorname{Im} \tilde{h}_i = \{y \in \mathrm{E}_A(k) \mid \mathfrak{m} \cdot y = 0\} = \operatorname{Im} \iota_{\mathfrak{m}}$$

and so $h_i := \iota_{\mathfrak{m}}^{-1} \cdot \tilde{h}_i$ is a well defined non-zero k-linear map $l_i \longrightarrow k$. We leave it to the reader to check that with these choices diagram (17) commutes. The last assertion follows from the commutativity of (17) and the fact that $\sum_{i=1}^{s} F^{semi}_{j_{\mathfrak{n}_i}}$ is an isomorphism by Proposition 3.8. □

We complete now the set $\{j_{\mathfrak{n}_1}, \dots, j_{\mathfrak{n}_s}\}$ of Sublemma 6.12 above to a family of injective hull injections $\underline{j} = (j_{(0)}, j_{\mathfrak{n}_1}, \dots, j_{\mathfrak{n}_s})$ for B as follows:

(i) If $K \neq L$ we let $j_{(0)} : L \xrightarrow{\simeq} (I'_0)_{(0)}$ be an arbitrary isomorphism.
(ii) If $K = L$ we let $j_{(0)}$ be the composition

$$B_{(0)} = K \xrightarrow{\iota_{(0)}} (I_0)_{(0)} \xrightarrow{(\zeta_0)^{-1}_{(0)}} \operatorname{Hom}_A(B, I_0)_{(0)} = (I'_0)_{(0)}.$$

Note that this is well defined since $(\zeta_0)_{(0)}$ is in this case an isomorphism.

Since the natural homomorphism $\mathrm{E}_B(l_i) \longrightarrow \mathrm{E}_B(l_i)_{\mathfrak{n}_i} \simeq \mathrm{E}_{B_{\mathfrak{n}_i}}(l_i)$ is an isomorphism and $(I'_0)_{(0)} = ((I'_0)_{\mathfrak{n}_i})_{(0)}$ we can consider $\underline{j}_i := (j_{(0)}, j_{\mathfrak{n}_i})$ as a family of injective hull injections for $B_{\mathfrak{n}_i}$ for all $1 \leq i \leq s$.

Sublemma 6.13. *There exists $0 \neq h \in \operatorname{Hom}_K(L, K)$ such that the following diagram commutes:*

$$\begin{tikzcd}
W(L) \arrow[r, "F_{j_{(0)}}"] \arrow[d, "\mathrm{cor}^h_{L/K}"] & \tilde{W}^0(L, (I'_\bullet)_{(0)}) \arrow[d, "\mathrm{Tr}_{B_{(0)}/A_{(0)}}"] & W^0(D^{(0)}_{B,\mu_{I'}}/D^{(1)}_{B,\mu_{I'}}, I'_\bullet) \arrow[l, "\mathrm{loc}^0_B"'] \arrow[r, "\partial^0_B"] \arrow[d, "\mathrm{Tr}_{B/A}"] & W^1(D^{(1)}_{B,\mu_{I'}}, I'_\bullet) \arrow[d, "\mathrm{Tr}_{B/A}"] \\
W(K) \arrow[r, "F_{\iota_{(0)}}"] & \tilde{W}^0(K, (I_\bullet)_{(0)}) & W^0(D^{(0)}_{A,\mu_I}/D^{(1)}_{A,\mu_I}, I_\bullet) \arrow[l, "\mathrm{loc}^0_A"'] \arrow[r, "\partial^0_A"] & W^1(D^{(1)}_{A,\mu_I}, I_\bullet).
\end{tikzcd} \quad (19)$$

If $K = L$ we can choose h to be id_K.

Proof. Let $h : L \longrightarrow K$ be the following morphism:

$$L \xrightarrow[\simeq]{j_{(0)}} (I'_0)_{(0)} = \operatorname{Hom}_A(B, I_0)_{(0)} \xrightarrow{(\zeta_0)_{(0)}} (I_0)_{(0)} \xrightarrow[\simeq]{\iota^{-1}_{(0)}} K$$

(we consider here all modules as A-modules and use the fact that $S^{-1}B = L$ for $S = A \setminus \{0\}$). If $K = L$ this is id_K by definition of $j_{(0)}$. We leave it to the reader to check that with this choice the left hand square of (19) commutes. The square on the right hand side commutes by [12, Thm. 2.9] because the restriction of scalars functor respects the codimension by support filtration, and the commutativity of the square in the middle is obvious from the definition of the transfer map; see [13, Sect. 4]. □

Having now defined $\underline{j}_i$, h_i for $i = 1, \dots, s$, and h we calculate $d^0_{\underline{\iota}} \cdot \mathrm{cor}^h_{L/K}$. By definition, see (15), we have

$$d^0_{\underline{\iota}} = (F_{\iota_{\mathfrak{m}}})^{-1} \cdot \partial^0_A \cdot (\mathrm{loc}^0_A)^{-1} \cdot F_{\iota_{(0)}},$$

and so we get, from Sublemma 6.13 above,

$$d^0_{\underline{\iota}} \cdot \mathrm{cor}^h_{L/K} = (F_{\iota_{\mathfrak{m}}})^{-1} \cdot \mathrm{Tr}_{B/A} \cdot \partial^0_B \cdot (\mathrm{loc}^0_B)^{-1} \cdot F_{j_{(0)}}. \quad (20)$$

Let $q_i : B \longrightarrow B_{\mathfrak{n}_i}$ be the localization morphism for $i = 1, \ldots, s$. By 5.10 the following diagram commutes:

$$\begin{tikzcd}
W^0(D^{(0)}_{B,\mu_{I'}}/D^{(1)}_{B,\mu_{I'}}, I'_\bullet) \arrow[r, "\partial^0_B"] \arrow[d, "q_i^*"'] & W^1(D^{(1)}_{B,\mu_{I'}}, I'_\bullet) \arrow[d, "q_i^*"] \\
W^0(D^{(0)}_{B_{\mathfrak{n}_i},\mu_{I'_{\mathfrak{n}_i}}}/D^{(1)}_{B_{\mathfrak{n}_i},\mu_{I'_{\mathfrak{n}_i}}}, (I'_\bullet)_{\mathfrak{n}_i}) \arrow[r, "\partial^0_{B_{\mathfrak{n}_i}}"] & W^1(D^{(1)}_{B_{\mathfrak{n}_i},\mu_{I'_{\mathfrak{n}_i}}}, (I'_\bullet)_{\mathfrak{n}_i}),
\end{tikzcd}$$

and hence we have since by definition (cf. Section 5) $\mathrm{loc}^1_B = (q_i^*)_{i=1}^s$:

$$\mathrm{loc}^1_B \cdot \partial^0_B \;=\; \left(\partial^0_{B_{\mathfrak{n}_i}} \cdot q_i^*\right)_{i=1}^s. \tag{21}$$

On the other hand the diagram

$$\begin{tikzcd}
W^0(D^{(0)}_{B,\mu_{I'}}/D^{(1)}_{B,\mu_{I'}}, I'_\bullet) \arrow[dd, "q_i^*"'] \arrow[dr, "\mathrm{loc}^0_B"] & \\
& W^0(L, (I'_\bullet)_{(0)}) \\
W^0(D^{(0)}_{B_{\mathfrak{n}_i},\mu_{I'_{\mathfrak{n}_i}}}/D^{(1)}_{B_{\mathfrak{n}_i},\mu_{I'_{\mathfrak{n}_i}}}, (I'_\bullet)_{\mathfrak{n}_i}) \arrow[ur, "\simeq", "\mathrm{loc}^0_{B_{\mathfrak{n}_i}}"'] &
\end{tikzcd}$$

commutes for all $1 \le i \le s$. Since by Theorem 5.2 the homomorphism $\mathrm{loc}^0_{B_{\mathfrak{n}_i}}$ is an isomorphism, we get from this and (21)

$$\partial^0_B \;=\; (\mathrm{loc}^1_B)^{-1} \cdot \left(\partial^0_{B_{\mathfrak{n}_i}} \cdot (\mathrm{loc}^0_{B_{\mathfrak{n}_i}})^{-1}\right)_{i=1}^s \cdot \mathrm{loc}^0_B.$$

Inserting this in Eq. (20) gives

$$\begin{aligned}
d^0_{\underline{\iota}} \cdot \mathrm{cor}^h_{L/K} &= (F_{\iota_{\mathfrak{m}}})^{-1} \cdot \mathrm{Tr}_{B/A} \cdot (\mathrm{loc}^1_B)^{-1} \cdot \\
&\qquad \cdot \left(\partial^0_{B_{\mathfrak{n}_i}} \cdot (\mathrm{loc}^0_{B_{\mathfrak{n}_i}})^{-1}\right)_{i=1}^s \cdot F_{j_{(0)}} \\
&= \left(\sum_{i=1}^s \mathrm{cor}^{h_i}_{l_i/k}\right) \cdot \left(\sum_{i=1}^s F^{semi}_{j_{\mathfrak{n}_i}}\right)^{-1} \cdot (\mathrm{loc}^1_B)^{-1} \cdot \\
&\qquad \cdot \left(\partial^0_{B_{\mathfrak{n}_i}} \cdot (\mathrm{loc}^0_{B_{\mathfrak{n}_i}})^{-1}\right)_{i=1}^s \cdot F_{j_{(0)}} && \text{by Sublemma 6.12} \\
&= \left(\sum_{i=1}^s \mathrm{cor}^{h_i}_{l_i/k}\right) \cdot \\
&\qquad \cdot \left((F_{j_{\mathfrak{n}_i}})^{-1} \cdot \partial^0_{B_{\mathfrak{n}_i}} \cdot (\mathrm{loc}^0_{B_{\mathfrak{n}_i}})^{-1} \cdot F_{j_{(0)}}\right)_{i=1}^s && \text{by (10)} \\
&= \sum_{i=1}^s \mathrm{cor}^{h_i}_{l_i/k} \cdot d^0_{\underline{j}_i} && \text{by (15).}
\end{aligned}$$

We are done. □

7. The graded Gersten–Witt complex

7.1. We have now all tools together to define the filtration by the powers of the fundamental ideal on the Gersten–Witt complex. Throughout this section X is a scheme with minimal dualizing complex $I_\bullet$ as in (12).

We fix a family $\underline{\iota} = (\iota_x)_{x \in X}$ of injective hull injections for X. Associated with this family we have the non-canonical Gersten–Witt complex $\mathrm{GW}(X, I_\bullet, \underline{\iota})$; cf. (14):

$$\bigoplus_{\mu_I(x)=m} W(k(x)) \xrightarrow{d_{\underline{\iota}}^m} \bigoplus_{\mu_I(x)=m+1} W(k(x)) \xrightarrow{d_{\underline{\iota}}^{m+1}} \cdots, \tag{22}$$

where $m = \min \mu_I$.

Lemma 7.2. *Let $y \in X_{\mu_I}^{(p)}$ and $x \in X_{\mu_I}^{(p+1)}$. Denote as $d_{\underline{\iota}}^{yx} : W(k(y)) \longrightarrow W(k(x))$ the yx-component of the differential $d_{\underline{\iota}}^p : \bigoplus_{\mu_I(y)=p} W(k(y)) \longrightarrow \bigoplus_{\mu_I(x)=p+1} W(k(x))$, and let $\kappa : \operatorname{Spec} \mathcal{O}_{X,x} \hookrightarrow X$ be the inclusion.*

(i) *If $x \notin \overline{\{y\}}$ then $d_{\underline{\iota}}^{yx} = 0$.*
(ii) *If $x \in \overline{\{y\}}$ then $P := \kappa^{-1}(y)$ is a prime ideal in $\mathcal{O}_{X,x}$ such that $A := \mathcal{O}_{X,x}/P$ is a one-dimensional local ring with dualizing complex, and $d_{\underline{\iota}}^{yx}$ is equal to a generalized second residue homomorphism $W(k(y)) \longrightarrow W(k(x))$ associated with a family of injective hull injections for A.*

Proof. This is a consequence of 5.9–5.11. □

From Theorem 6.6 we get the following

Corollary 7.3. $d_{\underline{\iota}}^p \left(\bigoplus_{\mu_I(y)=p} I^d(k(y)) \right) \subseteq \bigoplus_{\mu_I(x)=p+1} I^{d-1}(k(x))$ *for all $d \in \mathbb{Z}$.*

Therefore the complex (22) has the following subcomplexes $F^d\mathrm{GW}(X, I_\bullet, \underline{\iota})$:

$$\bigoplus_{\mu_I(x)=m} I^d(k(x)) \xrightarrow{d_{\underline{\iota}}^m} \bigoplus_{\mu_I(x)=m+1} I^{d-1}(k(x)) \xrightarrow{d_{\underline{\iota}}^{m+1}} \cdots. \tag{23}$$

We get a filtration of the Gersten–Witt complex $\mathrm{GW}(X, I_\bullet, \underline{\iota})$:

$$\mathrm{GW}(X, I_\bullet, \underline{\iota}) = F^0\mathrm{GW}(X, I_\bullet, \underline{\iota}) \supseteq F^1\mathrm{GW}(X, I_\bullet, \underline{\iota}) \supseteq F^2\mathrm{GW}(X, I_\bullet, \underline{\iota}) \supseteq \ldots$$

and short exact sequences of complexes

$$0 \longrightarrow F^{d+1}\mathrm{GW} \longrightarrow F^d\mathrm{GW} \longrightarrow F^d\mathrm{GW}/F^{d+1}\mathrm{GW} \longrightarrow 0 \tag{24}$$

for all $d \in \mathbb{Z}$, where we have set $F^i\mathrm{GW} = F^i\mathrm{GW}(X, I_\bullet, \underline{\iota})$.

Definition 7.4. Let X be a scheme with dualizing complex $I_\bullet$ and corresponding family of injective hull injections $\underline{\iota}$. We define for $d \in \mathbb{Z}$ the d-th *graded Gersten–Witt complex* of X to be the complex

$$\mathrm{GrGW}_d(X, I_\bullet) := F^d\mathrm{GW}(X, I_\bullet, \underline{\iota})/F^{d+1}\mathrm{GW}(X, I_\bullet, \underline{\iota}),$$

i.e. $\mathrm{GrGW}_d(X, I_\bullet) = 0$ if $d < 0$, and equal to the following complex:

$$\bigoplus_{\mu_I(x)=m} I^d(k(x))/I^{d+1}(k(x)) \xrightarrow{\mathfrak{d}_d^m} \bigoplus_{\mu_I(x)=m+1} I^{d-1}(k(x))/I^d(k(x)) \xrightarrow{\mathfrak{d}_d^{m+1}} \cdots$$

if $d \geq 0$, where the differential $\mathfrak{d}_d^p$ is induced by $d_{\underline{\iota}}^p$.

On the right hand side of the definition of $\mathrm{GrGW}_d(X, I_\bullet)$ there appears the family of injective hull injections $\underline{\iota}$, but on the left hand side it does not. This is justified by the following

Lemma 7.5. *Let $\underline{\iota}'$ be another family of injective hull injections for X. Then*

$$\operatorname{Im} d_{\underline{\iota}}^p(\alpha) - \operatorname{Im} d_{\underline{\iota}'}^p(\alpha) \in \bigoplus_{\mu_I(x)=p+1} I^{d-p}(k(x))$$

for all $\alpha \in \bigoplus_{\mu_I(x)=p} I^{d-p}(k(x))$, and all $m \leq p < n$ and $d \in \mathbb{Z}$. In particular, the induced differential $\mathfrak{d}_d^p : \mathrm{GrGW}_d(X, I_\bullet)^p \longrightarrow \mathrm{GrGW}_d(X, I_\bullet)^{p+1}$ does not depend on the chosen family of injective hull injections.

Proof. Let $y \in X^{(p)}_{\mu_I}$ and $x \in X^{(p+1)}_{\mu_I}$ such that $x \in \overline{\{y\}}$, and $\alpha \in I^{d-p}(k(y))$. Then we have for the yx-components $d^{yx}_{\underline{\iota}}$ and $d^{yx}_{\underline{\iota}'}$ of $d^p_{\underline{\iota}}$ and $d^p_{\underline{\iota}'}$:

$$\begin{aligned} d^{yx}_{\underline{\iota}}(\alpha) - d^{yx}_{\underline{\iota}'}(\alpha) &= F^{-1}_{\iota_x} d^p_X F_{\iota_y}(\alpha) - F^{-1}_{\iota'_x} d^p_X F_{\iota'_y}(\alpha) \\ &= F^{-1}_{\iota_x} d^p_X F_{\iota_y}(\alpha) - F^{-1}_{\iota_x} d^p_X F_{\iota'_y}(\alpha) \\ &\quad + F^{-1}_{\iota_x} d^p_X F_{\iota'_y}(\alpha) - F^{-1}_{\iota'_x} d^p_X F_{\iota'_y}(\alpha); \end{aligned}$$

see diagram (14). By Lemma 1.5 there exists $a \in \mathcal{O}^\times_{X,x}$ and $b \in \mathcal{O}^\times_{X,y}$ such that $F_{\iota'_x} = \langle a\rangle \otimes F_{\iota_x}$ and $F_{\iota'_y} = \langle b\rangle \otimes F_{\iota_y}$. Therefore

$$F^{-1}_{\iota_x} d^p_X F_{\iota_y}(\alpha) \; - \; F^{-1}_{\iota_x} d^p_X F_{\iota'_y}(\alpha) = d^{yx}_{\underline{\iota}}(\langle 1, -b\rangle \otimes \alpha)$$

and

$$F^{-1}_{\iota_x} d^p_X F_{\iota'_y}(\alpha) - F^{-1}_{\iota'_x} d^p_X F_{\iota'_y}(\alpha) = \langle 1, -a^{-1}\rangle \otimes d^{yx}_{\underline{\iota}}(\langle b\rangle \otimes \alpha)$$

since F_{ι_x} and F_{ι_y} are $W(\mathcal{O}_{X,x})$- and $W(\mathcal{O}_{X,y})$-linear, respectively. It follows that

$$d^{yx}_{\underline{\iota}}(\alpha) - d^{yx}_{\underline{\iota}'}(\alpha) = d^{yx}_{\underline{\iota}}(\langle 1, -b\rangle \otimes \alpha) \; + \; \langle 1, -a^{-1}\rangle \otimes d^{yx}_{\underline{\iota}}(\langle b\rangle \otimes \alpha),$$

and so we get the result from Corollary 7.3. $\square$

7.6. In this section all schemes are essentially of finite type over a field k, i.e. of finite type over k or a localization of such a finite type scheme.

We compare now our graded Gersten–Witt complex with Rost's [29] cycle cocomplex associated with the cycle module

$$k \longmapsto \operatorname{Gr}\bar{I}(k) := \bigoplus_{d\geq 0} I^d(k)/I^{d+1}(k).$$

It has been proven by Arason [1] that this is a cycle module in the sense of Rost [29]. Note that this means, in particular, that the transfer map of a finite field extension

$$\operatorname{cor}^h_{L/K} : I^d(L)/I^{d+1}(L) \longrightarrow I^d(K)/I^{d+1}(K)$$

does not depend on the choice of the non-zero K-linear map $h : L \longrightarrow K$.

We recall the definition of the cycle cocomplex. Let for this X be a scheme (essentially of finite type over a field k). The d-th cycle cocomplex $C_d(X, \operatorname{Gr}\bar{I})$ of the scheme X is the complex

$$\bigoplus_{x\in X^{(0)}} \bar{I}^d(k(x)) \xrightarrow{d^0} \bigoplus_{x\in X^{(1)}} \bar{I}^{d-1}(k(x)) \xrightarrow{d^1} \bigoplus_{x\in X^{(2)}} \bar{I}^{d-2}(k(x)) \xrightarrow{d^2} \cdots,$$

where $X^{(p)} \subseteq X$ is the set of points of codimension p in X, and we have set $\bar{I}^d(k(x)) := I^d(k(x))/I^{d+1}(k(x))$ for all $x \in X$. The differential

$$d^p \; : \; \bigoplus_{y\in X^{(p)}} \bar{I}^{d-p}(k(y)) \longrightarrow \bigoplus_{x\in X^{(p+1)}} \bar{I}^{d-p-1}(k(x))$$

is defined as follows. Let $y \in X^{(p)}$ and $x \in X^{(p+1)}$. Then the yx-component d^p_{yx} of d^p is zero if $x \notin \overline{\{y\}}$. If $x \in \overline{\{y\}}$ let $Z \longrightarrow \overline{\{y\}}$ be the normalization of the closure of y, and $x_1, \ldots, x_m \in Z$ the finitely points lying over $x \in \overline{\{y\}}$. They are all of codimension 1 in Z and therefore since Z is normal define second residue maps $\delta^{x_i} : \bar{I}^{d-p}(k(Z)) = \bar{I}^{d-p}(k(y)) \longrightarrow \bar{I}^{d-p-1}(k(x_i))$. Note that these maps do not depend on an uniformizer; cf. [1]. Since $Z \longrightarrow \overline{\{y\}}$ is a finite morphism the field extension $k(x_i)/k(x)$ is finite and so we have well defined transfer maps $\operatorname{cor}_{k(x_i)/k(x)} : \bar{I}^{d-p-1}(k(x_i)) \longrightarrow \bar{I}^{d-p-1}(k(x))$. Then by definition,

$$d^p_{yx} \; : \; \bar{I}^{d-p}(k(y)) \longrightarrow \bar{I}^{d-p-1}(k(x)) \qquad \alpha \longmapsto \sum_{i=1}^m \operatorname{cor}_{k(x_i)/k(x)}(\delta^{x_i}(\alpha)).$$

We get from the description of the differential of the graded Gersten–Witt complex in Lemma 7.2 and Propositions 6.5 and 6.10:

Theorem 7.7. *Let X be a scheme which is essentially of finite type over a field k, and $I_\bullet$ a dualizing complex of X (which exists; see Example 1.8 and Lemma 1.14) such that μ_I is the usual codimension, i.e. $\mu_I(x) = \dim \mathcal{O}_{X,x}$ for all $x \in X$. Then the d-th graded Gersten–Witt complex $\mathrm{GrGW}_d(X, I_\bullet)$ coincides with the d-th cycle cocomplex $C_d(X, \mathrm{Gr}\overline{I})$ of Rost [29].*

This has the following

Corollary 7.8. *Let $Y = \operatorname{Spec} \mathcal{O}_{X,x}$ be a local ring of a smooth variety over a field k, and $I_\bullet$ a minimal finite injective resolution of $\mathcal{O}_{X,x}$ considered as a dualizing complex of $\mathcal{O}_{X,x}$ as in Example 1.8. Then we have*

$$H^i(\mathrm{GrGW}_d(Y, I_\bullet)) = H^i(F^d\mathrm{GW}(Y, I_\bullet)) = 0$$

for all $i \geq 1$ and all $d \in \mathbb{Z}$.

Proof. This is known to be true by [29, Thm. 6.1] for $C_d(Y, \mathrm{Gr}\overline{I})$ and hence by the above theorem for $\mathrm{GrGW}_d(Y, I_\bullet)$. We deduce from this $H^i(F^d\mathrm{GW}(Y, I_\bullet)) = 0$ by induction on d, since we know it for $d = 0$ by [5, Thm. 4.3] or [14, Thm. 3.3], where another proof is given using coherent Witt theory. □

8. The Chow group of μ_I-codimension cycles

8.1. Let X be a scheme with dualizing complex $I_\bullet$. To state (and prove) Theorem 0.1 we have to define the Chow group of μ_I-codimension p-cycles. The definition is the same as for the usual Chow group, see Fulton [10], except that we replace codimension by μ_I.

Let

$$Z^p(X, \mu_I) := \bigoplus_{\mu_I(x)=p} \mathbb{Z}\,\overline{\{x\}},$$

be the free abelian group generated by μ_I-codimension p irreducible subschemes. To define rational equivalence let $x \in X^{(p)}_{\mu_I}$ and $y \in X^{(p-1)}_{\mu_I}$ with $x \in \overline{\{y\}}$. Since y is in the image of the inclusion $\kappa : \operatorname{Spec} \mathcal{O}_{X,x} \hookrightarrow X$ the local domain $A := \mathcal{O}_{X,x}/\kappa^{-1}(y)$ has dimension 1; see Lemma 1.14. Let $f \in k(y)^\times$. Then $f = \frac{g}{h}$ with $g, h \in A \setminus \{0\}$. We define the order of f at x to be the following integer:

$$\mathrm{ord}_x(f) := \mathrm{length}\, A/Ag - \mathrm{length}\, A/Ah$$

(this does not depend on the presentation of f as fraction; see [10, Lem. A2.5]), and we define the divisor of f with respect to the codimension function μ_I as

$$\mathrm{div}_{\mu_I}(f) := \sum_{x \in X^{(p)}_{\mu_I} \cap \overline{\{y\}}} \mathrm{ord}_x(f) \cdot \overline{\{x\}} \in Z^p(X, \mu_I)$$

(since by assumption X is noetherian, this is a finite sum). The subgroup of rational equivalence is then the group $\mathrm{rat}^p(X, \mu_I) \subseteq Z^p(X, \mu_I)$ generated by all elements $\mathrm{div}_{\mu_I}(f)$ with $f \in k(y)^\times$ and $y \in X^{(p-1)}_{\mu_I}$.

Definition 8.2. The quotient

$$\mathrm{CH}^p(X, \mu_I) := Z^p(X, \mu_I)/\mathrm{rat}^p(X, \mu_I)$$

is called the *Chow group of μ_I-codimension p-cycles*. We denote the class of an irreducible subscheme $Y = \overline{\{y\}}$ in $\mathrm{CH}^p(X, \mu_I)$ by $[Y]$.

Example 8.3. If X is a Cohen–Macaulay scheme with dualizing module Ω and $I_\bullet$ is a finite injective resolution of Ω considered as an element of $D^b_{coh}(\mathcal{M}(X))$ as in Example 1.8, then $\mu_I(x) = \dim \mathcal{O}_{X,x}$ for all $x \in X$, i.e. the codimension function of $I_\bullet$ is the usual codimension. Hence $\mathrm{CH}^p(X, \mu_I)$ is the usual Chow group of codimension p-cycles.

9. Proof of Theorems 0.1 and 0.2

9.1. Let X be a scheme with minimal dualizing complex $I_\bullet$. For ease of notation we assume that $\min \mu_I = 0$, i.e. $I_\bullet$ has the form

$$0 \longrightarrow I_0 \xrightarrow{d_0^I} I_{-1} \xrightarrow{d_{-1}^I} \cdots \xrightarrow{d_{-n+2}^I} I_{-n+1} \xrightarrow{d_{-n+1}^I} I_{-n} \longrightarrow 0,$$

where $n = \max \mu_I$.

There is a canonical surjective homomorphism

$$c_{X,\mu_I}^p \;:\; Z^p(X,\mu_I) \longrightarrow \bigoplus_{\mu_I(x)=p} W(k(x))/I(k(x))$$

which maps the cycle $\overline{\{y\}}$ to

$$<1> + \; I(k(y)) \in W(k(y))/I(k(y)) \subseteq \bigoplus_{\mu_I(x)=p} W(k(x))/I(k(x)).$$

Theorem 9.2. *The homomorphism c_{X,μ_I}^p induces an isomorphism*

$$\mathfrak{e}_{X,\mu_I}^p \;:\; \mathrm{CH}^p(X,\mu_I) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\simeq} H^p(\mathrm{GrGW}_p(X, I_\bullet))$$

for all $p \in \mathbb{Z}$.

Proof. If $p \le 0 = \min \mu_I$ there is nothing to prove and so let $p \ge 1$. Then we have

$$H^p(\mathrm{GrGW}_p(X,I_\bullet)) = \mathrm{Coker}\left(\bigoplus_{y \in X_{\mu_I}^{(p-1)}} I(k(y))/I^2(k(y)) \xrightarrow{\mathfrak{d}_p^{p-1}} \bigoplus_{x \in X_{\mu_I}^{(p)}} W(k(x))/I(k(x)) \right);$$

see Definition 7.4. To calculate this cokernel let $\alpha \in I(k(y))$ for some $y \in X_{\mu_I}^{(p-1)}$. We denote $\bar{\alpha}$ the class of α in $I(k(y))/I^2(k(y)) \subseteq \bigoplus_{y \in X_{\mu_I}^{(p-1)}} I(k(y))/I^2(k(y))$. The dimension indices $e_{k(x)}^0 : W(k(x)) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ (cf. Section 4) induce an isomorphism

$$\bigoplus_{x \in X_{\mu_I}^{(p)}} W(k(x))/I(k(x)) \xrightarrow{\simeq} \bigoplus_{x \in X_{\mu_I}^{(p)}} \mathbb{Z}/2\mathbb{Z} \simeq Z^p(X,\mu_I) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z},$$

which maps $\mathfrak{d}_p^{p-1}(\bar{\alpha})$ to the sum $\sum_{x \in X_{\mu_I}^{(p)} \cap \overline{\{y\}}} e_{k(x)}^0(\mathfrak{d}^{yx}(\bar{\alpha}))$, where $\mathfrak{d}^{yx}$ denotes the yx-component of $\mathfrak{d}_p^{p-1}$; cf. Lemma 7.2. For any field (of characteristic $\neq 2$) the fundamental ideal in the Witt ring is generated by 1-Pfister forms, and therefore the image of $\mathfrak{d}_p^{p-1}$ is generated by all elements $\mathfrak{d}^{yx}(\overline{\ll f \gg})$ with $f \in k(y)^\times$ and $y \in X_{\mu_I}^{(p-1)}$. On the other hand $\mathrm{rat}^p(X,\mu_I)$ is generated by all $\mathrm{div}_{\mu_I}(f)$ with $f \in k(y)^\times$ and $y \in X_{\mu_I}^{(p-1)}$ and so it is enough to show

$$e_{k(x)}^0(\mathfrak{d}^{yx}(\overline{\ll f \gg})) \;=\; \mathrm{ord}_x(f) + 2\mathbb{Z}$$

for all $f \in k(y)^*$, $y \in X_{\mu_I}^{(p-1)}$, and $x \in \overline{\{y\}} \cap X_{\mu_I}^{(p)}$.

We have

$$\ll \frac{g}{h} \gg \; + \; I^2(k(y)) = (\ll g \gg - \ll h \gg) + I^2(k(y))$$

in $I(k(y))/I^2(k(y))$ for all $g, h \in k(y)^*$ (cf. e.g. [30, Chap. 2, Sect. 11]) and so the result follows from Lemma 7.2 and the following one. $\square$

Lemma 9.3. *Let $(A, \mathfrak{m}, k)$ be a one-dimensional local domain with dualizing module Ω and quotient field K. Let further $f \in A \setminus \{0\}$. Then for all families $\underline{\iota} = (\iota_{(0)}, \iota_{\mathfrak{m}})$ of injective hull injections (corresponding to a minimal injective resolution of Ω) we have*

$$e^0_k(\,d^0_{\underline{\iota}}(\ll f \gg))\; =\; \text{length}\,(A/Af)\; + \mathbb{Z}/2\mathbb{Z},$$

where $d^0_{\underline{\iota}}$ is the generalized second residue map of Definition 6.2.

Proof. By Lemma 7.5 it is enough to prove this for a particular family $\underline{\iota} = (\iota_{(0)}, \iota_{\mathfrak{m}})$ of injective hull injections which we now choose. Let for this $I_0 \xrightarrow{d^I_0} I_{-1}$ be a minimal A-injective resolution of the dualizing module Ω. Since A is a domain we have $I_0 \simeq \mathrm{E}_A(A)$. We fix an embedding $\iota : A \hookrightarrow I_0 = \mathrm{Hom}_A(A, I_0)$ such that $\mathrm{Im}\,\iota \subset \mathrm{Ker}\,d_0 = \Omega$ (this is possible since $\dim \Omega = \dim A$). The localization of ι at the prime ideal (0) is an isomorphism $\iota_{(0)} : K \xrightarrow{\simeq} (I_0)_{(0)}$ which we choose as injective hull injection for the quotient field K. For $\iota_{\mathfrak{m}}$ we take any embedding $k \hookrightarrow \mathrm{E}_A(k)$. Consider now the following morphism of complexes:

$$\begin{array}{ccccccc} A: & \cdots \longrightarrow & A & \longrightarrow & 0 & \longrightarrow \cdots \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\iota} & & \downarrow & \\ \mathfrak{D}_I(A): & \cdots \longrightarrow & \mathrm{Hom}_A(A, I_0) & \xrightarrow{d^I_0} & \mathrm{Hom}_A(A, I_{-1}) & \longrightarrow \cdots, \end{array} \tag{25}$$

where we consider A as a complex concentrated in degree 0. The morphism ξ is symmetric with respect to the duality $\mathfrak{D}_I = \underline{R}\,\mathrm{Hom}_A(-, I_\bullet)$. Moreover, it is an isomorphism at the generic point, i.e. the $(A, \xi)_{(0)}$ is a 0-symmetric space in $D^b_{coh}(\mathcal{M}(K))$. We have

$$F_{\iota_{(0)}}(<1>) = [(A,\ \xi)_{(0)}] \qquad \text{and} \qquad F_{\iota_{(0)}}(<f>) = [(A,\ (-f)\cdot\xi)_{(0)}],$$

and hence: $e^0_k(\,d^0_{\underline{\iota}}(\ll f\gg)\,)\; =\; \chi(\,\mathrm{cone}\,[\,\xi\,\oplus\,(\,-f\cdot\xi\,)\,])$, where χ denotes the Euler characteristic; cf. (11). Since the morphism $(-f)\cdot\xi$ is the composite

$$A \xrightarrow{\cdot(-f)} A \xrightarrow{\xi} \mathfrak{D}_I(A)$$

the octahedron axiom [17, Chap. I, Section 1] implies the existence of an exact triangle

$$\mathrm{cone}(\cdot(-f)) \longrightarrow \mathrm{cone}\,\xi \longrightarrow \mathrm{cone}(-f\cdot\xi) \longrightarrow \mathrm{cone}(\cdot(-f))[1]\,,$$

where $\cdot(-f)$ denotes the morphism $A \xrightarrow{\cdot(-f)} A$ in $D^b_{coh}(\mathcal{M}(A))$ (considering A as an element of $D^b_{coh}(\mathcal{M}(A))$ concentrated in degree 0). Hence we get from Lemma 4.2:

$$\chi(\mathrm{cone}\;[\xi\oplus(-f\cdot\xi)]) = \chi(\mathrm{cone}\;[\cdot(-f)]) = \mathrm{length}(A/Af) \quad \mathrm{mod}\,2.$$

The lemma is proven. □

9.4. We get from (24) a long exact cohomology sequence. The composition of the connecting homomorphism $H^p(\mathrm{GrGW}_p(X, I_\bullet)) \xrightarrow{\partial} H^{p+1}(F^{p+1}\mathrm{GW}(X, I_\bullet, \underline{\iota}))$ with the natural morphism $H^{p+1}(F^{p+1}\mathrm{GW}(X, I_\bullet, \underline{\iota})) \longrightarrow H^{p+1}(\mathrm{GrGW}_{p+1}(X, I_\bullet))$ induces by Theorem 9.2 above a homomorphism

$$S^p_{X,I}\ :\ \mathrm{CH}^p(X, \mu_I) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}\ \longrightarrow\ \mathrm{CH}^{p+1}(X, \mu_I) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

If $X = \mathrm{Spec}\,A$ is an affine Cohen–Macaulay scheme we can give an explicit formula for this map. By [13, Thm. 6.2] a Cohen–Macaulay ring has a dualizing complex if and only if it has a dualizing (or also called canonical) module Ω. A minimal injective resolution $I_\bullet$ of Ω considered as an element of $D^b_{coh}(\mathcal{M}(A))$ as in Example 1.8 is then a dualizing complex of A for which we have $\mu_I(P) = \mathrm{ht}\,P$ for all $P \in \mathrm{Spec}\,A$. In particular, $\mathrm{CH}^p(A, \mu_I) = \mathrm{CH}^p(A)$ is the usual Chow group of codimension p-cycles. We set $S^p_{A,\Omega} := S^p_{A,I}$.

Theorem 9.5. *Let A be a Cohen–Macaulay ring with dualizing module Ω and P a prime ideal of height p. Then* $S^p_{A,\Omega}([\mathrm{Spec}\,A/P])\ =$

$$\sum_{P\subset Q,\,\mathrm{ht}\,Q=p+1} \mathrm{length}\left(\mathrm{Coker}(A/P \xrightarrow{\iota} \mathrm{Ext}^p_A(A/P, \Omega))_Q\right)\ [\mathrm{Spec}\,A/Q]$$

modulo $2\mathrm{CH}^p(A)$*, where $\iota\ :\ A/P \hookrightarrow \mathrm{Ext}^p_A(A/P, \Omega)$ is an arbitrary embedding.*

Proof. Let $\Omega \longrightarrow I_0 \xrightarrow{d_0^I} I_{-1} \xrightarrow{d_{-1}^I} \cdots \xrightarrow{d_{-n-1}^I} I_{-n}$ be a minimal A-injective resolution of the dualizing module Ω, where $n = \dim A$. By [12, Lem. 3.3] we have $\mathrm{Hom}_A(A/P, I_{-s}) = 0$ for $0 \le s < p$ and so

$$\mathrm{Ext}^p_A(A/P, \Omega) = \mathrm{Ker}(\,\mathrm{Hom}_A(A/P, I_{-p}) \xrightarrow{\bar{d}_{-p}} \mathrm{Hom}_A(A/P, I_{-p-1})),$$

where we have set $\bar{d}_{-p} = \mathrm{Hom}_A(A/P, d^I_{-p})$. In particular, we have $\mathrm{Ext}^p_A(A/P, \Omega) \subseteq \mathrm{Hom}_A(A/P, I_{-p}) = \mathrm{E}_{A/P}(A/P)$ and so P is an associated prime ideal of the A-module $\mathrm{Ext}^p_A(A/P, \Omega)$; cf. [8, Thm. 3.2.6]. It follows that there exists an injection $\iota : A/P \hookrightarrow \mathrm{Ext}^p_A(A/P, \Omega) \subseteq \mathrm{Hom}_A(A/P, I_{-p})$.

Consider now the following morphism of complexes:

$$\begin{tikzcd}
A/P \ : \arrow[d, "\xi"'] & 0 \arrow[r] \arrow[d] & A/P \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
\mathfrak{D}_{I_\bullet}(A/P)[p] \ : & 0 \arrow[r] & \mathrm{Hom}_A(A/P, I_{-p}) \arrow[r] & \mathrm{Hom}_A(A/P, I_{-p-1}).
\end{tikzcd}$$

This morphism is p-symmetric. Localizing ξ at a height p prime ideal shows that the pair (A, ξ) represents an element α of $W^p(D^{(p)}_{A,\mu_I}/D^{(p+1)}_{A,\mu_I}) \simeq \bigoplus_{\mathrm{ht}\, Q=p} W(k(Q))$ which in turn represents $\mathfrak{e}^p_{A,\mu_I}([A/P])$. The differential of the Gersten–Witt complex is given by the cone construction, see [3, Thm. 2.6], and therefore we have

$$S^p_{A,\Omega}([\mathrm{Spec}\, A/P]) = \sum_{P\subset Q,\, \mathrm{ht}\, Q=p+1} \chi(\mathrm{cone}(\xi)_Q)\, [\mathrm{Spec}\, A/Q] \ + \ 2\mathrm{CH}^{p+1}(A),$$

where (as above) χ is the Euler characteristic. The cone of ξ is the following complex:

$$A/P \xrightarrow{\ \iota\ } \mathrm{Hom}_A(A/P, I_{-p}) \xrightarrow{\bar{d}_{-p}} \mathrm{Hom}_A(A/P, I_{-p-1}) \longrightarrow \cdots.$$

Let $Q \supset P$ be a prime ideal of height $(p+1)$ in A. The ring $(A/P)_Q$ is a one-dimensional domain and so is Cohen–Macaulay. Moreover, we have $(I_{-r})_Q = 0$ if $r \ge p+2$ by [8, Lem. 3.2.5], and hence the complex

$$\mathrm{Hom}_A(A/P, I_{-p})_Q \xrightarrow{(\bar{d}_{-p})_Q} \mathrm{Hom}_A(A/P, I_{-p-1})_Q$$

is a dualizing complex of $(A/P)_Q$; see Proposition 1.10. By [13, Thms. 6.2 and 6.3] it follows that $(A/P)_Q$ also has a dualizing module which is given by $\mathrm{Ext}^p_A(A/P, \Omega)_Q$. In particular, we have $\mathrm{Coker}(\bar{d}_{-p})_Q = 0$ and $\mathrm{Ker}(\bar{d}_{-p})_Q = \mathrm{Ext}^p_A(A/P, \Omega)_Q$. Therefore the complex $\mathrm{cone}(\xi)_Q$ is quasi-isomorphic to the complex

$$(A/P)_Q \xrightarrow{\iota_Q} \mathrm{Ext}^p_A(A/P, \Omega)_Q,$$

and the result follows. □

10. The graded Gersten–Witt spectral sequence

10.1. Let as above X be a scheme with minimal dualizing complex $I_\bullet$ such that $\min \mu_I = 0$, and let $\underline{\iota}$ be a fixed family of injective hull injections for X. We set (as above)

$$F^d\mathrm{GW} := F^d\mathrm{GW}(X, I_\bullet, \underline{\iota}) \quad \text{and} \quad \mathrm{GrGW}_d(X, I_\bullet) := F^d\mathrm{GW}/F^{d+1}\mathrm{GW}.$$

Recall that by Lemma 7.5 the differential of the quotient complex $\mathrm{GrGW}_d(X, I_\bullet)$ does not depend on the choice of $\underline{\iota}$.

Associated with the short exact sequences of complexes

$$0 \longrightarrow F^{d+1}\mathrm{GW} \longrightarrow F^d\mathrm{GW} \longrightarrow \mathrm{GrGW}_d(X, I_\bullet) \longrightarrow 0$$

we have long exact cohomology sequences

$$\cdots \longrightarrow H^i(F^d\mathrm{GW}) \longrightarrow H^i(\mathrm{GrGW}_d(X, I_\bullet)) \xrightarrow{\ \partial\ }$$
$$H^{i+1}(F^{d+1}\mathrm{GW}) \longrightarrow H^{i+1}(F^d\mathrm{GW}) \longrightarrow \cdots.$$

These sequences constitute an exact couple and therefore give rise to a spectral sequence:

$$\mathrm{GrE}_1^{p,q}(X, I_\bullet, \underline{\iota}) := H^{p+q}(\mathrm{GrGW}_p(X, I_\bullet)),$$

the *graded Gersten–Witt spectral sequence* (note that the terms of the spectral sequence do not depend on $\underline{\iota}$ but the differentials may do). In other words, this is the spectral sequence associated with the filtration of the Gersten–Witt complex by the powers of the fundamental ideal; cf. [20, Chap. XI, Thm. 3.1].

10.2. If X has a real point, i.e. there is an $x \in X$ such that $-1 \in k(x)$ is a sum of squares, then the filtration $F^*\mathrm{GW}(X, I_\bullet, \underline{\iota})$ is not finite and hence the graded spectral sequence does not give much information about the homology of the Gersten–Witt complex. However, we have the following result. To state it we introduce some notation and recall the Milnor conjecture which we use in the proof. Let k be a field (as always in this work, of characteristic $\neq 2$), $H^*(k, \mathbb{Z}/2\mathbb{Z})$, the Galois cohomology of k with coefficients in $\mathbb{Z}/2\mathbb{Z}$, and $K_*^M(k)$ the Milnor k-theory of k. The Milnor conjecture [23] claims that there are natural isomorphisms

$$K_d^M(k)/2 \xrightarrow{\simeq} H^d(k, \mathbb{Z}/2\mathbb{Z}) \qquad \text{and} \qquad K_d^M(k)/2 \xrightarrow{\simeq} I^d(k)/I^{d+1}(k),$$

for all $d \geq 0$. Both conjectures have been recently proven by Voevodsky [33, Thm. 7.4] (the isomorphism between Galois cohomology and Milnor K-theory) and Orlov–Vishik–Voevodsky [25, Thm. 4.1] (the other one). For the last isomorphism $K_d^M(k)/2 \simeq I^d(k)/I^{d+1}(k)$ there is also another proof by Morel [24].

Proposition 10.3. *Let X be a scheme which is essentially of finite type over a field k, i.e. X is of finite type over k or a localization of such a scheme. Let $I_\bullet$ be a dualizing complex of X (which exists; see Example 1.8). If k has finite cohomological 2-dimension, then the graded Gersten–Witt spectral sequence converges to the cohomology of the Gersten–Witt complex:*

$$\mathrm{GrE}_1^{p,q}(X, I_\bullet, \underline{\iota}) \Longrightarrow H^{p+q}(\mathrm{GW}(X, I_\bullet, \underline{\iota})).$$

Proof. By [20, Chap. XI, Prop. 3.3] the spectral sequence of a filtered complex converges to the cohomology of this complex if the filtration is finite. It is essentially a consequence of the Milnor conjecture that the filtration of the Gersten–Witt complex is finite in the situation of our proposition.

Since k has finite cohomological 2-dimension, the same is true for the residue fields $k(x)$ of X by [31, Chap. II, Props. 10 and 11]. More precisely we have

$$H^d(k(x), \mathbb{Z}/2\mathbb{Z}) = 0 \qquad \text{for all } x \in X \quad \text{and all } d \geq N + 1 + \dim X,$$

where N denotes the cohomological 2-dimension of k. By the Milnor conjecture this implies that $I^d(k(x))/I^{d+1}(k(x)) = 0$ for all $x \in X$ and integers $d \geq N+1+\dim X$. Since by the Hauptsatz of Arason–Pfister [2] we have $\bigcap_{d=0}^{\infty} I^d(k(x)) = 0$ for all residue fields $k(x)$, it follows that

$$I^{N+1+\dim X}(k(x)) = 0 \qquad \text{for all } x \in X,$$

and we are done. □

Example 10.4. Examples of fields k which fulfill the condition of Proposition 10.3 above are: local fields, totally imaginary number fields, finite fields, and algebraically closed fields, or more general fields which satisfy property C_1; see [31, Chap. II] for proofs and more information.

10.5. *An easy application.* Another example of such a field is a 2-closed field k. This means that any finite algebraic extension of k has odd degree. Obviously any finite algebraic extension of k is then 2-closed, too, and therefore for all finite algebraic extensions L/k we have $I(L) = 0$.

Let now X be a regular integral scheme of dimension 4 which is essentially of finite type over the 2-closed field k. Let $I_\bullet$ be a minimal injective resolution of the structure bundle $\mathcal{O}_X$ of X. By the above remarks and Theorem 9.2 we have then a surjection $\mathrm{CH}^4(X)/2 \longrightarrow H^4(\mathrm{GW}(X, I_\bullet))$ (note that in this case μ_I is the usual codimension). On the other hand by [6, Thm. 10.1] there is also a surjection

$$H^4(\mathrm{GW}(X, I_\bullet)) \longrightarrow \mathrm{Ker}\,(W(X) \longrightarrow W(K)),$$

where K denotes the function field of X. We get the following generalization of a result due to Pardon [26] if X is affine and k is algebraically closed.

Proposition 10.6. *Let X be a integral regular scheme essentially of finite type over a 2-closed field k with function field $K = k(X)$. If $\dim X = 4$ then there is a surjection*

$$\mathrm{CH}^4(X)/2\mathrm{CH}^4(X) \twoheadrightarrow \mathrm{Ker}\,(W(X) \longrightarrow W(K))\,.$$

In particular, $W(X) \longrightarrow W(K)$ is injective if $\mathrm{CH}^4(X)$ is 2-divisible.

10.7. *Relation with the Bloch–Ogus spectral sequence.* Let in the following X be a smooth scheme over a field k, and $I_\bullet$ be a minimal injective resolution of the structure sheaf $\mathcal{O}_X$ of X. Then μ_I is the usual codimension, i.e. $\mu_I(x) = \dim \mathcal{O}_{X,x}$ for all $x \in X$. We set $\mathrm{GrGW}_d(X) := \mathrm{GrGW}_d(X, I_\bullet)$, and denote (as in Section 7.6) by $X^{(p)} \subseteq X$ the points of codimension p.

The *unramified Witt group* $W_{\mathrm{nr}}(X)$ of X is by definition the kernel of the 0-th differential ∂^0_X of the Gersten–Witt complex (2), i.e. $W_{\mathrm{nr}}(X) := H^0(\mathrm{GW}(X, I_\bullet, \iota))$. We define $I^d_{\mathrm{nr}}(X) := I^d(k(X)) \cap W_{\mathrm{nr}}(X)$. Denote further by $\mathcal{H}^d$ the (Zariski-)sheaf associated with the presheaf $U \longmapsto H^d_{\mathrm{et}}(U, \mathbb{Z}/2\mathbb{Z})$.

As shown by Voevodsky [33] and Orlov–Vishik–Voevodsky [25] (cf. also Morel [24]), the Milnor [23] homomorphism

$$e_k \ : \ \mathrm{Gr}\bar{I}(k) := \bigoplus_{d\geq 0} \bar{I}^d(k) \ \longrightarrow \ \bigoplus_{d\geq 0} H^d(k, \mathbb{Z}/2\mathbb{Z}),$$

where we have set $\bar{I}^d(k) = I^d(k)/I^{d+1}(k)$, is well defined and an isomorphism for all fields k. We get from this, see Parimala [28, Sect. 1], a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I^d_{\mathrm{nr}}(X)/I^{d+1}_{\mathrm{nr}}(X) \arrow[r] \arrow[d, "\simeq"] & \bar{I}^d(k(X)) \arrow[r, "d^0"] \arrow[d, "e_{k(X)}"'] & \bigoplus_{d\geq 0} \bar{I}^{d-1}(k(x)) \arrow[d, "(e_{k(x)})_{x\in X^{(1)}}"] \\
0 \arrow[r] & \Gamma(X, \mathcal{H}^d) \arrow[r] & H^d(k(X), \mathbb{Z}/2\mathbb{Z}) \arrow[r] & \bigoplus_{d\geq 0} H^{d-1}(k(x), \mathbb{Z}/2\mathbb{Z}).
\end{tikzcd}$$

Here d^0 is induced by second residue homomorphisms and therefore coincides with the 0-th differential of the graded Gersten–Witt complex $\mathrm{GrGW}_d(X)$, and the lower row is the complex of global sections of the Bloch–Ogus [7] resolution of $\mathcal{H}^d$. Hence it follows from Corollary 7.8 that the complex of sheaves $U \longmapsto \mathrm{GrGW}_d(U)$ is a flasque resolution of $\mathcal{H}^d$, too. Therefore we have isomorphisms

$$H^{p+q}(\mathrm{GrGW}_p(X)) \ \simeq \ H^{p+q}(X, \mathcal{H}^p)$$

for all $p, q \in \mathbb{Z}$. In particular, the E_2-terms of the Bloch–Ogus [7] spectral sequence coincide with our graded Gersten–Witt spectral sequence. However, the differentials of these two spectral sequences go in different directions.

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